17th Mediterranean Conference on Control & Automation
Makedonia Palace, Thessaloniki, Greece
June 24 - 26, 2009

A Novel Energy Pumping Strategy for Robotic Swinging

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***Abstract* – In this work we show that an Acrobot can be made to behave as a robotic swing. This is achieved by controlling the first joint, provided that a given condition is satisfied. When this condition is not satisfied, the system undergoes through singular points. Even when this happens, we are again able to make the system behave as a swing by controlling the second joint and employing a new Energy Pumping strategy. This strategy presents important advantages compared to previously proposed strategies, as it is the only one that can start the system from rest and drive it to large heights. Moreover, it is fast and requires very small torques.**

***Index Terms* – Robotic swing, energy pumping, gymnast robots, underactuated systems.**

I. Introduction

The swing problem has attracted the interest of a number of researchers during the last thirty years. Indeed, the problem is very interesting as it involves increasing the energy of a multibody system using internal (chemical in the case of humans) energy or motions. However, most research focused on dynamic analysis rather than on methods that can result in robotic swinging using controls. Also, none of these studied the effects of singular points or verified if the proposed movements are feasible for a particular under-actuated system.

The work up-to-date can be classified in two broad categories. The first deals with swing analysis and reports alternative kinematic strategies without a plan to implement them with active control. They also focus on techniques that allow an increase of the width of oscillation of a system (for Energy Pumping) but do not deal with how to make the system swing with a given amplitude and keep this oscillation constant, see for example [1-7]. One of the earliest works considered the swing model as a simple pendulum with variable length, [1]. Several years later, a strategies for initiation and pumping the swing from a standing position was published following a qualitative approach [2]. Swinging from standing and sitting positions was studied and it was concluded that the swing is best characterized as a forced oscillator, [3], [4]. Two different kinds of swinging were compared in [5]. In another study, the question whether people act as self optimizing machines while they swing was investigated, [6]. These studies do not address the issue of robotic swinging, which is dealt with in [7], using a sitting swing strategy but relying only on linear controls based purely on common experience.

The second type of work deals with the *Acrobot* problem in which the goal is to bring the system (an underactuated inverted pendulum) to the up right position, [8-12]. In his pioneering work, M. Spong used partial feedback linearization to bring the Acrobot to the upright position, [9]. Later, researchers tried to achieve the same goal, but most controllers were based on energy methods (e.g. [10], [11]). Other works have used Lyapunov methods and were successful to bring the first Acrobot link to any desired position [12]. Bringing the Acrobot to the up right position with constraints to the second link has been studied, [13]. This kind of motion is close to the motion that gymnasts make on the high bar.

The aim of this paper is to study robotic swinging of an Acrobot-type robot using partial model based control. Here, the second link is restricted from making a full revolution. The encountered singular points due to the loss of angular momentum coupling are studied. Their dynamic nature is explained, as well as how they can be avoided using a new swinging strategy. A new energy pumping strategy is proposed that presents important advantages over existing strategies. This strategy can start the system from rest, is fast and requires low torques.

II. System Dynamics

To study the robotic swing and the pumping of energy that occurs, (i.e. the transfer of energy from the actuated dof to the unactuated one), an Acrobot-type system is employed. The Acrobot is an under-actuated robotic system with two degrees of freedom, (dof), i.e. the angle of the first link, q_1, and the angle of the second link, q_2, see Fig. 1. Of those, only the second dof is actuated.

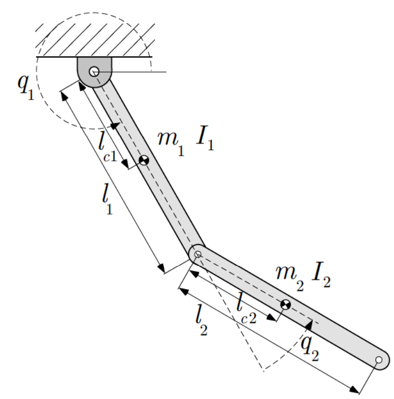

Figure 1. Acrobot system and its parameters.

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Since the structure of this robotic system approximates a sitting person swinging, it was chosen as the system to be studied here.

The equations of motion may be derived using the Lagrangian of the system and are described by,

$$\mathbf{M(q)\ddot{q}} + \mathbf{C(q,\dot{q})\dot{q}} + \mathbf{G(q)} = \boldsymbol{\tau} \tag{1}$$

where $\mathbf{q} = [q_1, q_2]^T$, $\boldsymbol{\tau} = [0, \tau_2]^T$, and $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{G}$ are given in Appendix A.

In this paper, we are interested in designing a controller capable of initiating a swinging motion of the system and converging to a swinging oscillation with given amplitude. Since the robotic swing is underactuated, one can directly control one of the two degrees of freedom only. As it will be presented later, a dual strategy is chosen to swing the robotic system. If we understand the swing system well enough to produce a close to ideal energy pumping strategy, then the convenient second degree of freedom (q_2) is used as our controlled variable. However, if no such strategy is available, then the first degree of freedom (q_1) is controlled under the requirement to oscillate such that the entire system behaves like a swing. In both cases, we use partial model-based control with nonlinear terms cancellation.

III. Robotic Swing with Control on the First Joint

Swinging when q_1 is controlled is facilitated by the coupling terms in (1). Due to this fact, no special strategy is needed to initiate swinging, and this is clearly an advantage. Since the first joint is not actuated, its motion must be generated by the actuator acting on the second joint.

Another advantage is that employing control on q_1 and studying the resulting response of q_2, allows one to develop a new strategy for swinging and Energy Pumping. However, a disadvantage of using control on q_1 is the appearance of singular points. When these occur, the system is unable to pump energy smoothly, and as a consequence, the required torque gets large accelerating the second link. Next, we develop the swinging strategy with control on q_1 and start with planning.

1) *Planning*

Here q_1 is controlled. We are interested in initiating swinging, and upon reaching a desired amplitude, to be able to hold the motion so that the system at the steady state swings. A simple strategy is to require that q_1 changes as a sinusoidal function with continuously increasing width of oscillation,

$$q_{1d}(t) = q_{1offset} + C_1 \cdot \sin(\omega t) \tag{2}$$

where,

$$C_1 = \begin{cases} k \cdot t & \text{for} \quad t_f > t \geq 0 \\ k \cdot t_f = C_{1f} & \text{for} \quad t \geq t_f \end{cases} \tag{3}$$

where k is the constant rate at which the oscillation amplitude increases, t_f is the time at which the steady state oscillation occurs, C_{1f} is the width of oscillation at the steady state, and ω is the lower natural frequency of the system, computed around the stable equilibrium point of the robot; this enables the system to swing, requiring small torque. This strategy works even if the input frequency is not equal to the lower natural frequency of the system. However, in such a case, a higher torque will be needed.

In (2), $q_{1offset}$ determines the angle around which the oscillation occurs. For swinging and the conventions in Fig. 1, $q_{1offset}$ is $270°$. Fig. 2 shows the oscillation of the first joint around its offset value. The response can be divided in two different states,

(a) The transient state, where Energy Pumping occurs and the width of oscillation continuously increases, and,

(b) The steady state, where Energy Pumping does not occur and the width of oscillation remains constant.

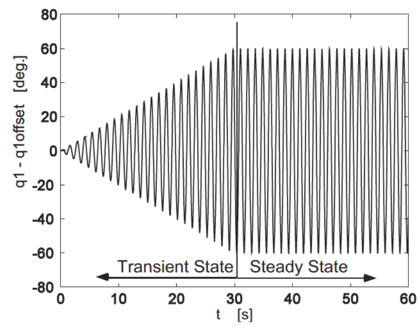

Figure 2. The transient and steady state response for the first joint angle.

Although (2) and (3) are very simple, they still allow one to define both the t_f, the width of oscillation C_1 and the speed at which this is reached, k. Obviously, these parameters have an effect on required actuator torques and size.

2) *Model Based Control*

Here, our aim is to force the system to follow the trajectory described by (2). This can be done using a partial Model Based Control technique with nonlinear term cancellation.

To do this, a second order differential equation, with respect to q_1, that contains the input torque is needed. This can be obtained using the equations of motion, (1). By eliminating $\ddot{q}_2$ from (1), we come up to the following:

$$\ddot{q}_1 \cdot A_1 + B_1 = \tau_2 \tag{4}$$

where:

$$A_1 = \frac{g_1(q_1, q_2)}{-(p_2 + p_3 \cdot \cos q_2)} \tag{5}$$

$$B_1 = \frac{f_1(q_1, q_2, \dot{q}_1, \dot{q}_2)}{-(p_2 + p_3 \cdot \cos q_2)} \tag{6}$$

where functions g_1, f_1, p_2 and p_3, are given in Appendix A.

The following controller makes sure that q_1 will reach its desired value in prescribed time,

$$\tau_2 = \left(\ddot{q}_{1d} + k_p \cdot (q_{1d} - q_1) + k_d \cdot (\dot{q}_{1d} - \dot{q}_1)\right) \cdot A_1 + B_1 \tag{7}$$

Equation (7) constitutes a Model Based Control with nonlinear term cancellation. Assuming knowledge of system parameters, terms A_1, and B_1 cancel the nonlinear terms in (4), while the terms in the parentheses constitute a PD feedback controller that can regulate the system response using the control gains k_p, k_d.

3) Singular points

Looking carefully at (5), (6) and (7) one can easily see that the denominator may become equal to zero. This point to the existence of algorithmic singular points. These have no relationship to kinematics, and cannot be computed using the Jacobian of the system. Their location depends on system physical parameters. In addition, generally these points appear only during the transient state.

In order to investigate the effects of these points, we set as K the denominator in question and study it further.

$$K = p_2 + p_3 \cos(q_2) \tag{8}$$

As it was mentioned before, when a system comes from a singular point, then a denominator is becoming equal to zero and the controller fails. To obtain a clear physical meaning of what happens at such points, we find the system angular momentum with respect to the first joint. This is given by,

$$\begin{aligned} H &= \left(p_1 + p_2 + 2p_3\cos(q_2)\right)\cdot \dot{q}_1 + \left(p_2 + p_3\cos(q_2)\right)\dot{q}_2 \\ &= \left(p_1 + p_2 + 2p_3\cos(q_2)\right)\cdot \dot{q}_1 + K\cdot \dot{q}_2 \end{aligned} \tag{9}$$

The angular momentum is constituted of two terms. The first term is the contribution of the first link and the second is the contribution of the second one. Since at singular points K is zero, it can be seen that at such points the second link has no effect on system angular momentum, and the coupling, which is important for energy pumping, is lost.

Singular point existence causes problems to system behavior. At such instances, the response of q_1 is not smooth any more, and the torque τ_2 locally increases drastically, trying to reduce the tracking error in q_1. Since no coupling exists at these points, the torque rapidly accelerates the second link, making it to undergo full rotations. In such cases, pumping of energy is erratic and no proper swinging can result. Despite this, swinging may occur, but this may take unpredictable time.

The important question that arises is whether it is possible to design a controller capable of swinging without requiring large torques and without unacceptably high accelerations of the second link. To this end, we examine when the term K can be nonzero.

$$K = p_2 + p_3\cos(q_2) > 0 \Rightarrow p_2 > p_3 \tag{10}$$

Substituting the terms p_2, and p_3, (10) becomes.

$$m_2 l_{c2}^2 + I_2 > m_2 l_1 l_{c2} \tag{11}$$

Using the expression for I_2 given in Appendix A, (11) becomes:

$$l_2 > \frac{3}{2}\cdot l_1 \tag{12}$$

If (12) holds, then coupling between the two links never fails and pumping can occur without infinite torques and second link accelerations.

4) New Energy Pumping strategy

Up to this point, swinging and energy pumping is possible only if (12) is in effect. An important question is whether it is possible to develop a new Energy Pumping strategy, which could provide sufficient pumping, without going through singular points, and even if (12) is not in effect.

Notice that singular points appear due to the exploitation of the coupling between the two links and drive the first joint using the actuator for the second joint. Therefore, to avoid the singularities, it is natural to explore the possibility of driving the second joint directly. Based on this observation, our aim is to find a new strategy of Energy Pumping that can be used to pump energy in systems in which (12) does not hold.

A new strategy can be developed influenced by the study of the response of q_2. In order to do this, we study simulation results obtained using a system in which (12) does *not* hold and therefore the second link is not always coupled dynamically to the first one. This is motivated by the fact that despite the non smooth response of the system, after long time, the system tends to stabilize in some smooth swinging. This is shown in Figure 3, where the system has an erratic behavior for about 26 s, but swings after that time.

We define the following variables,

$$\begin{aligned} q_{1new}(t) &= q_1(t + a\cdot t_f),\ a > 1 \\ q_{2new}(t) &= q_2(t + a\cdot t_f),\ a > 1 \end{aligned} \tag{13}$$

that describe the system response during *smooth swinging*.

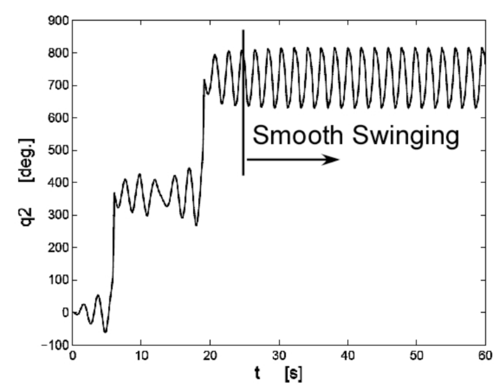

Figure 3. The response of the second angle reaches eventually a steady state and smooth swinging.

To learn from q_{1new} and q_{2new} orbits, we record the smooth swinging response part and analyze it with the help of Fourier analysis. Applying an FFT algorithm on the steady state part of the response of q_2, see Figure 4, one can notice: (a) the appearance of peaks at higher harmonics of the input frequency, and (b) that the energy of the first harmonic is by far the highest. This observation allows us to neglect the higher harmonics and keep the first one only. This points to the direction that q_1 oscillates with relatively large amplitude, when q_2 is a pure sinusoidal function with a single frequency, close to the lowest natural frequency, and has a constant difference in phase from q_1.

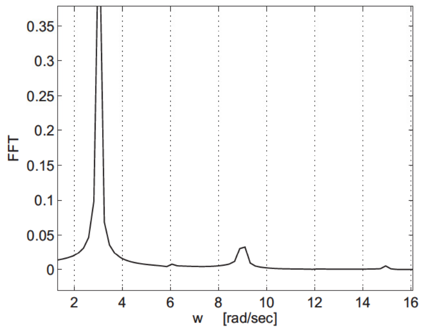

Figure 4. Frequencies contained in q_{2new} response (after the transient).

The phase difference in question can be found by studying the response of q_2, with q_1 controlled and (12) not in effect. In cases where the actual phase difference deviates from this value, then the system might still be capable of Energy Pumping but will require a higher torque. This however can only be achieved provided the input frequency is close to the natural frequency.

We can now proceed with the development of a new strategy for Energy Pumping, i.e. we determine how the second link q_2 should move so that the unactuated first angle q_1 increases its width of oscillation. Based on the previous observations, Energy Pumping can occur if the second angle is driven by

$$q_{2d}(t) = C_2 \sin(\omega t + \varphi) \tag{14}$$

where ω is the lowest system natural frequency, and φ is a phase difference between q_1 and q_2.

The advantage of this pumping strategy over others is that it can start with *zero* initial conditions and result in large oscillation amplitudes. Although this strategy does not maintain constant amplitude of oscillation, and therefore it is not a strategy for swinging, it is still a new strategy for effective Energy Pumping and can be used to increase the width of oscillation of a robotic swing.

IV. Robotic swing with Control on the Second Joint

The advantage of using control on q_2 is that is very easy to be controlled since q_2 is the actuated degree of freedom. As mentioned earlier, the disadvantage is it requires a good swing strategy. This is discussed next.

1) *Planning*

The system must be able to swing at desired amplitude. Therefore, during the transient response, a pumping strategy is needed. When the desired level of swinging is reached, pumping must stop. This is achieved by the following command for q_2,

$$q_{2d}(t) = \begin{cases} C_2 \cdot \sin(\omega t + \varphi) & \text{if } C_1 < C_{1f} \\ q_2^* = const. & \text{if } C_1 \geq C_{1f} \end{cases} \tag{15}$$

where q_2^* is the value of q_2 at the moment when q_1 reaches the desirable amplitude for the first time and the first link angular speed is null (for smoother switching). Upon examination of (15), one can easily see that to increase the width of oscillation, the Energy Pumping strategy developed earlier is used. To evaluate the performance of swinging, an amplitude error is defined as

$$e_a(i) = C_1(t_i) - C_{1f},\ t_i = \{t \,/\, \dot{q}_1(t_i) = 0 \wedge q_1(t_i) > 270°\} \tag{16}$$

which indicates the distance of the amplitude at zero velocity after the stable equilibrium point from the desired one. Once the correct amplitude is achieved, the second joint is locked and the system behaves as a simple pendulum. With this strategy, either the transient settling time or the oscillation amplitude can be set. Parameter C_2, which determines the maximum width of oscillation that the system can reach, is found by trial and error. In general, high values result in reduced oscillation amplitude accuracy.

2) *Model based control*

With a methodology similar to that in Section III, one can design a control law to force the system follow the desired trajectory. Following some manipulation of (1), we get,

$$\ddot{q}_2 \cdot A_2 + B_2 = \tau_2 \tag{17}$$

where A_2, B_2 are given in Appendix A and are functions of the states and velocities. To guarantee tracking for q_2, a partial model based control law with nonlinear term cancellation is designed that yields the torque τ_2 as,

$$\tau_2 = \left(\ddot{q}_{2d} + k_d \cdot (\dot{q}_{2d} - \dot{q}_2) + k_p \cdot (q_{2d} - q_2)\right) \cdot A_2 + B_2 \tag{18}$$

V. Simulation Results

In this section, we first assume a system in which (12) applies and by controlling the first joint, (controller in Section III), we make it swing and realize pumping of energy. Next, we study a system for which (12) does not hold, and in which singular points exist. Using phase information from this system, we apply the controller of Section IV on the second joint and show that this results in smooth swinging and energy pumping.

A. *Model-based Control on q_1*

Table I displays the parameters of a system in which condition (12) holds. Therefore, no singular points are expected while controlling the first joint. The system starts with null initial conditions, the settling time is chosen to be 30 s, and the final width of oscillation of the first link, $C_{1f} = 60°$. Then, (3) yields,

$$k = 0.03489 \text{ rad/s} \tag{19}$$

Table I. Parameters of a system for which (12) holds.

m_1 [kg]	m_2 [kg]	l_1 [m]	l_2 [m]	ω [rad/s]
10.00	20.00	0.50	1.00	4.91

Figure 5 displays the joint angle responses and the applied torque on the second joint. The system response has a smooth and stable behavior. Angle q_1 follows the desired trajectory, and the error $e_1 = q_{1d} - q_1$, (not shown), is practically zero. Also, as shown from the response of q_2, the second link does not accelerate continuously and does not complete full rotations. Since no singularities exist, the input torque at the second joint is small and smooth.

Next, the same controller is used to initiate swinging, but here (12) does not hold. The system parameters are shown in Table II. The remaining conditions are as before. Figure 6 shows the system response and the applied torque. During the transient state, angle q_1 follows the desired trajectory with some small error, which disappears at the steady state. Nevertheless, the second link is accelerated by very large torques that try to compensate for the loss of coupling. The result is that the link undergoes full rotations and no swinging is achieved.

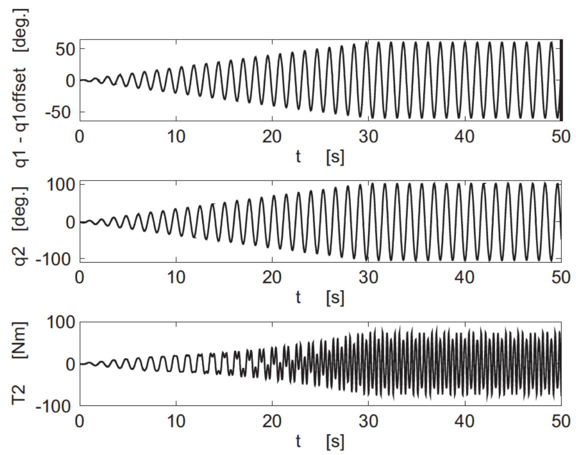

Figure 5. Response with control on q_1, without singular points.

TABLE II. PARAMETERS FOR A SYSTEM FOR WHICH (12) DOES NOT HOLD.

m_1 [kg]	m_2 [kg]	l_1 [m]	l_2 [m]	ω [rad/s]
10.00	5.00	0.50	0.25	4.22

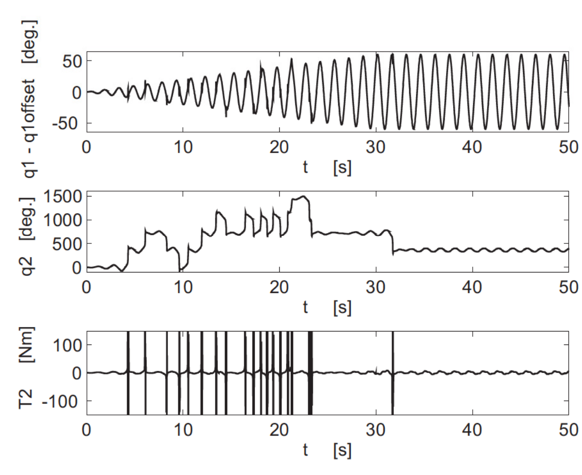

Figure 6. Response with control on q_1, in the presence of singular points.

Following the erratic transient phase, the system achieves a swinging response, see Figure 7. Fitting a sinusoidal functions on the response of the two angles, results in correlation coefficient very close to "1". From these, the difference in phase is found to be:

$$\varphi = 1.073\ rad = 61.5° \tag{20}$$

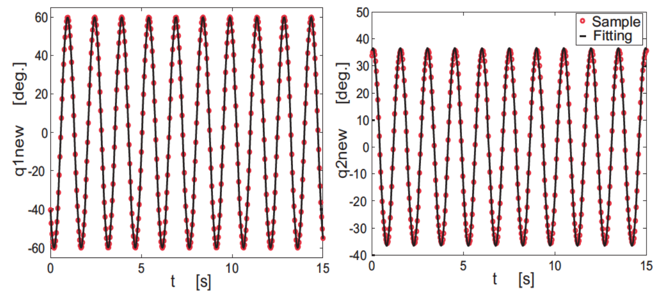

Figure 7. Response of q_{1new} and q_{2new} fitted with sinusoidal functions.

B. Model-based Control on q_2

We apply the strategy that was developed in Section IV, using a system whose parameters are given in Table II. The desired trajectory for q_2 is given by (14) and the phase difference is given by (20). The system starts from null initial conditions and $C_{1f} = 60°$. The parameter C_2 is set to 0.98 so that pumping is fast. Figure 8 displays the obtained system response. The response is smooth as desired. The system starts from null initial conditions and reaches the desired width of oscillation very quickly and with small amplitude errors. In addition the required torque is smooth and small in magnitude.

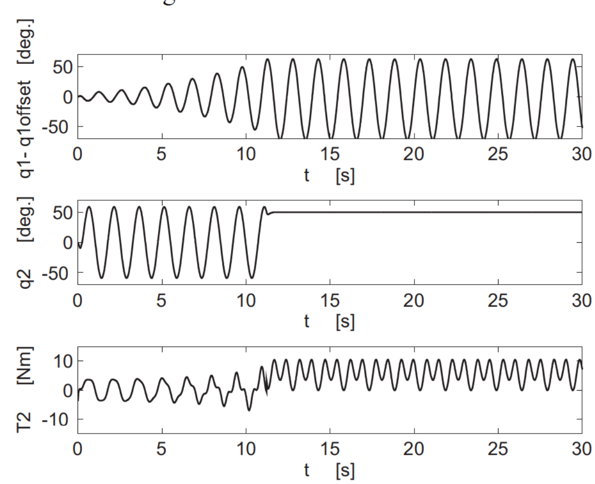

Figure 8. Response with control on q_2.

Here we emphasize that, to the best of our knowledge, the developed Energy Pumping strategy is the only which can start the system from zero initial conditions, and can lead to high swinging amplitudes in a controlled fashion.

As was shown above, with this Energy Pumping strategy, the system can swing smoothly. The developed method allows one to require energy pumping up to a specific settling time or energy pumping up to a given level. Setting both the height (amplitude) that the system will reach and the settling time requires further analysis. Also, the oscillation amplitude that is achieved is close but not equal to the desirable value. This is due to the strategy used to keep the width of oscillation constant and can be modified with increased complexity.

VI. Conclusions

In this work we showed that an Acrobot can be controlled to behave as a robotic swing. This can be achieved by controlling the first joint angle q_1 when a particular condition is satisfied. In this case, we can set the desirable oscillation amplitude, as well as the time in which this must be achieved. It was shown that this strategy may be subject to singular points depending on system parameters. These singular points result in large input torques, and high second link accelerations. When this condition is not satisfied, then again we are able to swing the system controlling q_2 and employing a new Energy Pumping strategy. This new strategy requires that the second link should have as an orbit a sinusoidal function with frequency equal to the lower natural frequency of the system. The difference of phase between the two orbits depends system parameters and is constant. This strategy presents important advantages compared to others since it is the only one that can begin with null initial conditions and make the system reach high swinging amplitudes. In addition, it is very fast and requires very small torques.

References

[1] Burns, J. A., "More on pumping a swing," *American Journal of Physics,* 1970, vol. 38, pp.920-922.

[2] Walker, J., "How to get the playground swing going a first lesson in the mechanics of rotation," *Scientific American*, March, 1989, pp. 106-109.

[3] Case, W. B., "The pumping of a swing from the standing position," *American Journal of Physics*, 1996, vol. 64, pp. 215-220.

[4] Case, W. B., and Swanson, M. A., "The pumping of a swing from the seated position," *American Journal of Physics*, vol. 58, 1990, pp. 463-467.

[5] Wirkus, S., Rand, R., and Ruina, A., "How to Pump a Swing," *The College Mathematical Journal*, September 1998, Vol. 29, No. 4, pp. 266-275.

[6] Piccoli B., and Kulkarni J., "Pumping a Swing by Standing and Squatting," *IEEE Control Systems Magazine*, August 2005, pp.48-56.

[7] Nipper N., Godowski J., Arroyo A., and Schwartz E., "Robotic Swing Drive as Exploit of Stiffness Control Implementation," *2001 Florida Conference on Recent Advances in Robotics*, Tallahassee, FLA, 2001.

[8] Fantoni, I, and Lozano, R., *Non-linear Control for Under-actuated Mechanical systems*, Springer, London, 2002.

[9] Spong, M., "The Swingup Control Problem for the Acrobot," *IEEE Control Systems Magazine,* Feb 1995, Vol. 15, No., pp.49-55.

[10] Banavar R., and Mahindrakar A., "Energy-based Swing-up of the Acrobot and Time-optimal Motion," *Proc. IEEE Conference on Control Applications (CCA '03)*, Istanbul, Turkey, June 23-25, 2003, pp. 706- 711.

[11] Mahindrakar A., and Banavar R., "A swing-up of the Acrobot based on a simple pendulum strategy," *International Journal of Control*, April 2005, Vol. 78, No. 6, pp. 424-429.

[12] Zergeloglu E., Dixon W., Dawson D., Jaffar S., and Hannan M., "Lyapunov-Based Set-Point Control of the Acrobot," *IEEE International Conference on Control Applications*, (CCA '98), Trieste, Italy, September 1998, pp. 887-891.

[13] Tomohiro, H., Takahiro, W., Mingcong D., Akira, I., Nobuyuki, U., and Hirashima, Y., "Swing-up Control of an Acrobot Having a Limited Range of Joint Angle of Two Links," *5th Asian Control Conference*, 2004, pp. 1071-1076.

Appendix A

The matrices and vectors $\mathbf{M}$, $\mathbf{C}$, and $\mathbf{G}$, are given by,

$$\begin{aligned}
\mathbf{M} &= \begin{pmatrix} p_1 + p_2 + 2p_3\cos(q_2) & p_2 + p_3\cos(q_2) \\ p_2 + p_3\cos(q_2) & p_2 \end{pmatrix} \\
\mathbf{C} &= \begin{pmatrix} -p_3\sin(q_2)\dot{q}_2 & -p_3\sin(q_2)\dot{q}_2 - p_3\sin(q_2)\dot{q}_1 \\ p_3\sin(q_2)\dot{q}_1 & 0 \end{pmatrix} \\
\mathbf{G} &= \begin{pmatrix} p_4 g\cos(q_1) + p_5 g\cos(q_1+q_2) \\ p_5 g\cos(q_1+q_2) \end{pmatrix}
\end{aligned} \tag{21}$$

with parameters:

$$\begin{aligned}
p_1 &= m_1 l_{c1}^2 + m_2 l_1^2 + I_1 \\
p_2 &= m_2 l_{c2}^2 + I_2 \\
p_3 &= m_2 l_1 l_{c2} \\
p_4 &= m_1 l_{c1} + m_2 l_1 \\
p_5 &= m_2 l_{c2}
\end{aligned} \tag{22}$$

$$\begin{aligned}
&I_1 = \frac{1}{12} m_1 (l_1)^2,\ I_2 = \frac{1}{12} m_2 (l_2)^2 \\
&l_{c1} = l_1/2,\ l_{c2} = l_2/2
\end{aligned} \tag{23}$$

The functions g_1 and f_1 are given by:

$$g_1 = p_2\cdot(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2) - (p_2 + p_3\cdot\cos q_2)^2 \tag{24}$$

$$\begin{aligned}
f_1 = &-p_3\cdot\dot{q}_1^2\cdot\sin q_2\cdot(p_2 + p_3\cdot\cos q_2) - p_3\cdot\dot{q}_2^2\cdot\sin q_2\cdot p_2 \\
&-2\cdot p_3\cdot p_2\cdot\dot{q}_1\dot{q}_2\cdot\sin q_2 - \\
&-p_5\cdot g\cdot\cos(q_1+q_2)\cdot(p_2 + p_3\cdot\cos q_2) + \\
&+p_4\cdot g\cdot\cos q_1\cdot p_2 + p_5\cdot g\cdot\cos(q_1+q_2)\cdot p_2
\end{aligned} \tag{25}$$

The functions A_2 and B_2 are given by:

$$A_2 = p_2 - \frac{(p_2 + p_3\cdot\cos q_2)^2}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} \tag{26}$$

$$\begin{aligned}
B_2 = &\frac{\dot{q}_1^{\,2}\cdot(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)\cdot p_3\cdot\sin q_2}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} + \\
&+\frac{p_3\cdot\dot{q}_2^{\,2}\cdot\sin q_2\cdot(p_2 + p_3\cdot\cos q_2)}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} + \\
&+\frac{+2\cdot p_3\cdot\dot{q}_1\cdot\dot{q}_2\cdot\sin q_2\cdot(p_2 + p_3\cdot\cos q_2)}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} + \\
&+\frac{-(p_2 + p_3\cdot\cos q_2)\cdot p_4\cdot g\cdot\cos q_1}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} + \\
&+\frac{p_5\cdot g\cdot\cos(q_1+q_2)\cdot p_3\cdot\cos q_2}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)} + \\
&+\frac{p_5\cdot g\cdot\cos(q_1+q_2)\cdot p_1}{(p_1 + p_2 + 2\cdot p_3\cdot\cos q_2)}
\end{aligned} \tag{27}$$