On Modeling and Control of a Holonomic Vectoring Tricopter

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Abstract—The modeling and control of a vectoring tricopter UAV are developed in this article. The UAV is actuated by three thrust motors, each guided by suitable actuators, thus forming a platform able to independently track any desired attitude and trajectory. The derivation of the equations of motion is followed by the development of a vectoring controller that is supplemented by an allocation strategy. Both are based on geometric feedback linearization techniques, resulting in a singularity-free control law, taking into account the inertia effects of the main body, of the motors, and of the vectoring dynamics (actuators). A stability proof is developed validating the effectiveness of the control strategy under bounded disturbances. Simulations showcase the developed controller and tricopter performance.

I. INTRODUCTION

Advances during the last decade in sensors, materials, electronics and power sources made the construction of small scale UAVs feasible, and resulted in a wave of research in this area. Small scale UAVs have a variety of industrial and military applications ranging from aerial photography, inspection and mapping, to surveillance and security. Substantial work has been devoted in quadrotor UAVs due to their simple construction and mechanics, resulting in vehicles able to perform aggressive and cooperative maneuvers, and grasping and transportation of objects [1],[2].

Although initial quadrotor designs were subject to axes coupling, currently the development of UAVs with independent axis control is given high priority since such capabilities can extend the use of UAVs as versatile field robots [3],[4],[5],[6]. Indeed, a UAV underactuated (uUAV) with respect to the main body degrees-of-freedom (dof), must tilt its body in order to move sideways limiting its ability to attain the proper pose that would allow it to traverse cluttered spaces. For inspection/surveillance tasks, uUAVs need a supplementary apparatus like controllable/actuated camera mounts along with elaborate path planing trajectories, so that the transmitted camera image can be kept level and not tire an operator. More difficult physical interaction tasks, such as force/torque application in arbitrary directions or assembly tasks cannot be performed by a uUAV but instead are only possible under some very specific conditions [2].

In contrast to a uUAV, a holonomic vehicle with independent axes control can vary independently its position and attitude, forming a true six dof robotic platform, capable for any inspection/surveillance task, for traversing cluttered space, and for assembly or interaction tasks, with no external apparatuses, keeping at the same time the overall size small.

Towards this direction, a UAV having eight rotors (quadrotor with four side rotors) was proposed aiming to decouple the attitude/translational dynamics [3]. However, the independent translational/attitude dynamics were restricted to a narrow envelope of the orientation. A hexrotor design with complete control over its trajectory/attitude was proposed in [4], and in [5] where the UAV has variable pitch propellers. However, the fixed orientation of the canted rotors results in poor maneuverability at some attitudes. A quadrotor with four tilting propellers independently spanning both the E3 and SO(3) spaces was proposed in [6]. This design still utilizes four thrusting rotors.

In view of the above, this work is motivated by the need to design a vehicle with independent axes control and with a small number of thrusting motors. Its focus is the development of a holonomic vehicle using only three vectoring thrusting rotors, i.e. less than those in [3],[4],[5],[6], and a computationally inexpensive control strategy that will not be susceptible to singular configurations, resulting in an UAV with decoupled translational/attitude response. The derivation of the equations of motion is done using a Newton-Euler approach that allows display of the internal forces and torques between the thrusting motors and their base. A geometric allocation strategy based on the complete multibody dynamics, and a geometric vectoring controller are developed and result in a singularity-free control law. This control law takes into account the full system dynamics, including the inertia effects of the main body, of the thrusting motors and of the vectoring actuators, and remains effective in the presence of modeling inaccuracies. It is shown by a stability proof that the control law results in a uniformly ultimately bounded system under bounded disturbances. Simulation results validate the developed controller and showcase responses characterized by independent motion in six dimensions.

II. DYNAMICS

A. Description and definitions. The vectoring tricopter UAV is comprised of three thrusting motors located at the extremities of the UAV’s three legs (see Fig. 1); each guided by suitable actuators capable to point each thrusting motor. The legs are coplanar with an 120° axial offset. The tricopter frame is chosen on overall cost considerations. The frame is simple in construction, requires less materials and is also lighter in contrast to a quad/hexarotor frame that has to support more motors. Furthermore its geometry allows for flexibility in placing payloads, such as a camera, closer to
the center of mass (CM) due to the 120° offset between the tricopter legs.

The vectoring definition implies that each motor will be pointed in $S^2$, i.e. the unit sphere in the three dimensional space. The actuator assembly together with the $i^{th}$ motor/propeller comprises the vectoring apparatus (VA) (see Fig. 1) that generates a thrusting force/gyroscopic moment on leg $i$. An inertial reference frame $I_R\{E_1, E_2, E_3\}$ and a base-fixed frame $I_b\{e_1, e_2, e_3\}$ at the base center of mass (CM) are chosen, together with three $i^{th}$ motor-fixed frames $I_i\{e_{i,1}, e_{i,2}, e_{i,3}\}$ where $i = 1, 2, 3$ denotes the $i^{th}$-VA.

![Fig. 1. Free-body diagram (FBD) of the Vectoring Tricopter concept with the coordinate frames, forces, moments and vectors that define it.](image)

**TABLE I. Definitions**

<table>
<thead>
<tr>
<th>Right Subscript</th>
<th>(,)_i</th>
<th>$i^{th}$-VA number</th>
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<tbody>
<tr>
<td>(,)_p</td>
<td>Desired rotation matrix/vector signal</td>
<td></td>
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<tr>
<td>$^{m}_i$</td>
<td>Vector expressed in fixed base frame $I_b$</td>
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<tr>
<td>$^{m}_p$</td>
<td>Vector expressed in $i^{th}$-motor-fixed frame, $I_i$</td>
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Definitions:

- $x, x_i \in \mathbb{R}^3$ Position of the base, $i^{th}$-VA wrt $I_R$ in $I_R$
- $v, v_i \in \mathbb{R}^3$ Velocity of the base, $i^{th}$-VA wrt $I_R$ in $I_R$
- $\omega \in \mathbb{R}^3$ Angular velocity of the base wrt $I_R$ in $I_b$
- $\omega_i \in \mathbb{R}^3$ Angular velocity, $i^{th}$-VA wrt $I_b$ in $I_i$
- $Q \in SO(3)$ Rotation matrix from $I_i$ to $I_b$ frame
- $Q_i \in SO(3)$ Rotation matrix from $I_i$ to $I_b$ frame
- $\omega_i = \sum_{i} J_{i}^{+} \omega_i$ Total gravity force in $I_i$ [10]
- $b_F, b_p \in \mathbb{R}$ Force and Torque constants
- $g_r, g_p \in \mathbb{R}$ Gravity and Air density constants
- $m \in \mathbb{R}^3$ Vector connecting the extremity of Leg $i$ with the CM of the $i^{th}$-VA in $I_i$
- $q_{i,1} \in \mathbb{R}^3$ Unit vector, co-linear to the $i^{th}$ propeller axis, equal to $Q_i(\hat{m}_p/\|\hat{m}_p\|)$ in $I_b$
- $p_i \in \mathbb{R}^3$ Vector connecting the CM of the base with the extremity of Leg $i$ in $I_i$
- $f_i \in \mathbb{R}$ Force component of the $i^{th}$-VA in the $\{e_{i,3}\}$ direction equal to $b_F(\hat{\omega}_i, e_{i,3})^2$ [10]
- $J \in \mathbb{R}^{3\times3}$ Inertial matrix (IM) of the base in $I_b$
- $J_i \in \mathbb{R}^{3\times3}$ IM of $i^{th}$-VA in $I_i$
- $m \in \mathbb{R}$ Total UV mass ($m_b + 3m_p$) where $m_b$: mass of the frame and $m_p$: mass of a VA
- $g_0 \in \mathbb{R}^3$ Gravity force ($-mg_0 E_3$) on VA, in $I_R$
- $g_0 \in \mathbb{R}^3$ Gravity force ($-mg_0 E_3$) on base, in $I_R$
- $g \in \mathbb{R}^3$ Total gravity force ($g_0 + 3g_p$) on UAV
- $bF_D \in \mathbb{R}^3$ Wind disturbance force in $I_b$ [11]
- $bM_D \in \mathbb{R}^3$ Wind disturbance moment in $I_b$ [11]

Note that if the left superscript of a vector is omitted, then the vector is expressed in $I_b$. The VA is modeled as a rigid body attached on leg $i$, see Fig. 1. During pointing the VA and simultaneously maintaining a desired propeller speed about the pointing direction, a force $f_e e_{i,3}$ [10] and a torque $b_T f_i e_{i,3}$ [1] are generated. Thus the third component of the control torque $^{m}_u_i$ is the torque of the $i^{th}$ thrusting propeller while the first two components are required to point the $i^{th}$ motor. Finally the UAV configuration is defined by the location of its base CM, $x$ and base attitude, $Q$ together with the attitude of the three VA’s, $Q_i, i=1,2,3$. The configuration manifold is $G = SO(3) \times SO(3) \times SO(3)$.

**B. Kinetics.** The Newton-Euler methodology is employed for deriving the UAV equations of motion. The position of the CM of the $i^{th}$-VA is,

$$b_x = Q^T x_i, x_i = x + Q(p_i + m_p b u_i)$$

The $i^{th}$-vectoring apparatus dynamics are described by,

$$b_F = Q_i f_i e_{i,3}$$

$$bM = (Q_i(-m_p)) \times bF + M_{c,i} + M_{p,i}$$

where $bF, bM_{c,i}, bM_{p,i}$ are forces and resulting moments applied on the $i^{th}$-VA by the base and given by,

$$bF_{c,i} = m_p b v_i - Q^T g_p$$

$$bM_{c,i} = (Q_i(-m_p)) \times bF_{c,i}, m_p = de_{i,3} = [0; 0; d]$$

The cross product mappings $S()$, $S^{-1}()$, the accelerations, and the angular velocities are defined in the Appendix.

Having the dynamic equations of the VA, the equations for the translational and attitude dynamics of the UAV can now be derived. Following the Newton-Euler methodology, for the translational dynamics, see “Leg” in Fig. 1, the motion can be described by the tricopter frame/base dynamics under the influence of the forces $-bF_{c,i}$ generated by each VA (calculated by (1a)) and the body-exerted gravity force $g_0$. Moment-wise (see “Leg” in Fig. 1) each VA exerts a moment $-m_p u_i$, on the base. Then, the $base$ dynamics are given by,

$$\dot{x} = v$$

$$m_b v + Q^T (bF_D - \sum_{i=1}^{3} bF_{e,i})$$

$$J^T \hat{\omega} = \sum_{i=1}^{3} \left(-Q_i m u_i + b_p i \times (-bF_{e,i}) \right)$$

$$\hat{Q} = QS(\hat{\omega})$$

The disturbance wrench $[bF_D, bM_D]$, simulating the wind is assumed to be bounded and is modeled as in [11]. The inputs for the equations above are the pointing moments $m_p u_i (i=1,2,3)$ that actuate the $i^{th}$-VA, pointing it in space while producing the motor generated thrust forces $bF_{p,i}$ and moments $bM_{p,i}$. Note that the equations of motion are highly coupled. This can be seen by rewriting the equations in
matrix form with respect to \([v; \omega; \dot{\omega}; m\omega_1; m\omega_2; m\omega_3]\),
\[
M(Q, Q_1, Q_2, Q_3) = \begin{bmatrix}
  \dot{v} \\
  b\omega \\
  m\omega_1 \\
  m\omega_2 \\
  m\omega_3
\end{bmatrix} + 
\begin{bmatrix}
  Q(\sum_{i=1}^{3} \Phi_i - \Delta) \\
  E \\
  \Gamma_1 \\
  \Gamma_2 \\
  \Gamma_3
\end{bmatrix} = U \tag{5}
\]

\(\dot{Q} = QS(\omega), \ \dot{Q} = QS(m\omega_i), i = 1, 2, 3\)
where \(M \in \mathbb{R}^{15 \times 15}\) see (AS) in the last page and,
\[
U = \begin{bmatrix}
  0; \sum_{i=1}^{3} (-Q_i m u_i); m u_1; m u_2; m u_3
\end{bmatrix}
\]
\[
\Phi_i = b\omega \times (b\omega \times (b\rho_i + Q m p)) + 2(b\omega \times Q_i (m\omega_1; m\omega_2; m\omega_3))
\]
\[
\Delta = g_b + Q \Theta_F + \sum_{i=1}^{3} (F_p i + T g_p + g_p)
\]
\[
E = \sum_{i=1}^{3} S(b\rho_i) m p_i - b\rho_i \times (Q g_p + b F_p i)
\]
\[
\Gamma_i = m \omega_i \times J_p m \omega_i - Q \Theta F_S(Q(-m p)) m p_i - B_i
\]
\[
B_i = Q T [M p_i - (Q(-m p)) \times (Q g_{p} - b F_p i)]
\]
The equations of motion of the entire UAV are given by (1-4), or in matrix form by (5).

III. CONTROL DESIGN

Since the goal is the design of a holonomic vehicle able to track arbitrary poses, a suitable controller is needed with an almost global operational envelope. This condition guides us to employ geometric control techniques as they are singularity free and simplify the control design [7],[8],[9],[14].

First we propose a geometric thrust allocation strategy that produces the reference pointing directions and propeller velocities for each VA so that the base is able to track a desired pose. Then we design a controller able to point each VA with respect to the base, while simultaneously actuating the VA’s propeller to the desired speed. This task is carried out assuming that the forces and moments transmitted by the VA’s to the base can be estimated approximately using sensors and/or computation, allowing us to treat the VA’s as separate systems. Finally a stability proof is developed showing that with the developed controller, the vehicle exhibits holonomic response tracking a desired pose under bounded disturbances.

A. Allocation strategy: We develop a tricopter thrust allocation strategy, using the attitude error function \(\Psi(Q, Q_d)\), see (A1). This attitude error function captures the current attitude as it is encapsulated in \(Q\) with the desired attitude given by \(Q_d\). If \(Q\) is antipodal to \(Q_d\), using (A1) then \(\Psi = 2\), translating to the maximum attitude difference of 180° with respect to an equivalent axis angle rotation. If \(Q = Q_d\) then \(\Psi = 0\) signifying the same attitude. This attitude error function yields the attitude/angular velocity errors defined as in [8],
\[
e_{\omega, Q} = \left[ \begin{array}{c}
e_{\omega} \\
e_Q
\end{array} \right] = \left[ \begin{array}{c}
\frac{x - x_d}{2\sqrt{1 + tr(Q_d Q)} S^{-1}} (Q_d^T Q - Q T Q_d) \\
x_Q
\end{array} \right] \tag{6}
\]
\[
e_{\omega} = [e_q, e_{\omega}] = [v - v_d; b\omega - Q T Q_d b\omega] \tag{7}
\]
error vectors. As a result the controller in [7],[9] is subject to the limitation that the angular velocity is driven to zero. Since we track time-varying attitudes and non-zero angular velocities, those configuration errors are unsuitable.

Instead we use $\Psi_i$ as an intermediate step to find $b_{e,q,i}$, $b_{e,\omega,i}$ for time varying $b_{q_i,d}, \omega_{i,d}$ to get $b_{e,q,i}=b_{q_i,d} \times b_{q_i}, b_{e,\omega,i}=b_{\omega,i} \times b_{q_i,d}$. Note that in the intermediate angular velocity error $b_{e,\omega,i}$, both $b_{\omega,i}$ and $\omega_{i,d}$ lie on different tangent spaces. To correct this, we transform the intermediate error vectors in $I_k$ to derive the final form of the $i^{th}$-VA configuration errors as,

$$m_{e,q,i} = Q_i^T (b_{q_i,d} \times b_{q_i})$$

(14)

$$m_{e,\omega,i} = Q_i^T (b_{\omega,i} - b_{\omega,i,d}) = m_{\omega,i} - Q_i^T Q_i d m_{\omega,i,d}$$

which are consistent on the $S_1^q, T_{q_i}S_{q_i}^2$ manifold error vectors.

Their derivatives are calculated by differentiating (14),

$$m_{\dot{e},q,i} = Q_i^T (b_{\dot{q}_i,d} \times b_{q_i,d} \times b_{q_i}) - Q_i^T (b_{\dot{q}_i,d} \times b_{q_i} - b_{\dot{q}_i,d} \times b_{\omega,i})$$

(15)

$$m_{\dot{e},\omega,i} = m_{\dot{\omega},i} + S^T (m_{\omega,i}) Q_i d m_{\omega,i,d} - Q_i^T Q_i d m_{\omega,i,d}$$

to effectively negotiate the model/sensor inaccuracies while attaining any allocation generated pointing direction, a geometric sliding mode methodology will be employed on $S_1^q$. The sliding surfaces are constructed in terms of $\Psi_i$, $\Psi_{i+d}$ and its associated configuration and velocity errors in order to get a Lyapunov function written in terms of $\Psi_i$. Then the control design is similar to nonlinear control design in Euclidean spaces. The defined sliding surfaces and their derivatives are,

$$S_i = (\Lambda + \Psi_i)^m e_{q,i} + \gamma^m e_{\omega,i}$$

(16)

$$\dot{S}_i = \dot{\Psi}_i e_{q,i} + (\Lambda + \Psi_i)^m \dot{e}_{q,i} + \gamma^m \dot{e}_{\omega,i}$$

where $\Lambda > 0$ and $\gamma > 0$ are positive gains. The chosen Lyapunov functions and their derivatives are,

$$V_i = (S_i^T S_i)/2, \dot{V}_i = S_i^T S_i$$

To avoid chattering, the convergence of all system trajectories to the sliding surface will be realized by choosing the control $u_i$ such that when not on the surface the following holds,

$$V_i = S_i^T S_i = \sum_{j=1}^n s_{i,j}s_{i,j} \leq -\kappa \|S_i\|^2$$

(17)

where $\kappa>0$ and the subscript $(.)_j$ signifies component wise manipulations. The derived control law is,

$$m_{u,i} = \eta^{-1} J_p (\eta^m \alpha_i - \eta \dot{f}_i - \dot{\Psi}_i m_{e,q,i}) - (\Lambda + \Psi_i)^m e_{q,i} - \gamma S_i$$

(18a)

$$m_{\alpha,i} = Q_i^T Q_i d m_{\omega,i,d} - S^T (m_{\omega,i}) Q_i d m_{\omega,i,d}$$

(18b)

$$f_i = J_p^{-1} (Q_i^T (\Lambda + b_{\omega,i,d}) - m_{\omega,i,d} - J_p m_{\omega,i})$$

(18c)

where $\gamma > 0$, while $(.)$ signifies parameter identification errors. Using (18a) and (1b) in (17), after some manipulations,

$$s_{i,j} \dot{s}_{i,j} = -\gamma \left( J_p \right)_{j,j} + \gamma \left( J_p \right)_{j,j} - \gamma \left( J_p \right)_{j,j}$$

(19a)

$$T_{i,j} = \left( J_p^{-1} J_p - 1 \right)_{j,j} \left( \eta^m \alpha_i - (\Lambda + \Psi_i)^m \dot{e}_{q,i} - \dot{\Psi}_i m_{e,q,i} - \eta \dot{f}_i \right)_j$$

(19b)

Observing the above equations, we note that as $\gamma$ is increased, the trajectories converge faster to the sliding surfaces, while the error boundary around the desired response shrinks. It can be shown that the derived control law is singularity-free, smooth (no high frequency chattering, see Fig. 2(h)), it can handle bounded modeling inaccuracies, and it stabilizes each VA around any given equilibrium except the antipodal equilibrium.

This is the best that can be achieved, since $b_{q_i,d} \times b_{q_i}$ vanishes at the antipodal equilibrium (almost global). The case were the allocation strategy generates the antipodal equilibrium $b_{q_i,d} = -b_{q_i}$, will make the $i^{th}$-VA stay at $b_{q_i}$, was studied through extensive simulations which show that, due to the VA’s inability to change attitude, an infinitesimal change to the UAV’s configuration will take place. Thus in the next control iteration the desired attitude $b_{q_i,d}(t+dt)$ will not be the unstable equilibrium. This case could be triggered also if the allocation strategy receives as command a dramatic change of the UAVs attitude equilibrium. This can be avoided by choosing a proper reference trajectory.

IV. STABILITY ANALYSIS

A stability analysis for the attitude dynamics is developed first, followed by a stability analysis for the position dynamics. A necessary condition for the proofs to hold is that the system lives in $L_2$={($Q_i \in SO(3)$)||$Q_i Q_i^T < 2$}, meaning, the requested orientation is not antipodal to the current one.

A. Attitude Stability. The attitude error dynamics are calculated by differentiating the attitude component of (7), namely $e_{\omega}$. Substituting into the resulting expression (4b), followed by (1b) solved for $m_{u,i}$, and (3a) to get,

$${\dot{J}}e_{\omega} = J \left( S^T (b_{\omega}) Q_i^T b_{\omega,d} - Q_i^T Q_i d b_{\omega,d} \right)$$

$$+ \sum_{i=1}^3 \left( 3 A_i \times [Q_i^T b_{\omega,i} - m_p b_{\dot{\omega},i}] - Q_i J_p (e_{\omega,i} + m_{\alpha,i}) - Q_i (m_{\omega,i} \times J_p m_{\omega,i}) + A_M [b_{F_{p,1}}; b_{F_{p,2}}; b_{F_{p,3}}] + b_{M_D} \right)$$

(20)

Substituting $b_{F_{p,i}} = b_{F_{p,i}} + b_{F_{e,p,i}}$ to (20) and the attitude component of (8) we get,

$${\dot{J}}e_{\omega} = b_{\omega,D} + \sum_{i=1}^3 \left( -Q_i J_p e_{\omega,i} + A_i \times [-m_p b_{\dot{\omega},i}] \right)$$

(21)

$$+ A_M [b_{F_{e,p,1}}; b_{F_{e,p,2}}; b_{F_{e,p,3}}] + b_{M_D}$$

The final form of the attitude error dynamics is,

$${\dot{J}}e_{\omega} = -k_R e_{\omega} - k_Q e_{\omega} - c_{2Q} e_{\omega} - k_{\omega} e_{\omega} + \epsilon_Q$$

(22)

$$\epsilon_Q = \sum_{i=1}^3 \left( A_i \times [-m_p b_{\dot{\omega},i}] - Q_i J_p e_{\omega,i} \right)$$

where $e_{\omega}$ is the attitude disturbance term that includes VA tracking errors, wind disturbances and VA related accelerations ($\dot{\dot{\omega}}_i$) see (A2). Due to the small mass/inertia properties of the VA these are dealt as disturbances. Our choice of attitude/angular velocity tracking errors dictates a Lyapunov function as in [8], with $c_{2Q} > 0$,

$$V_Q = (e_{\omega,0} \cdot J e_{\omega,0})/2 + k_R \Psi(Q, Q_d) + c_{2Q} e_{\omega,0} \cdot e_{\omega}$$

(23)

It can be shown that the following inequality holds see [8],

$$\lambda_{min}(W_11) \|z\|^2 \leq V_Q \leq \lambda_{max}(W_112) \|z\|^2$$

(24)
where $z = [\|e_Q\|; \|e_!\|] \in \mathbb{R}^2$ and $W_{11}, W_{12}$ are given by,

\[
W_{11} = \frac{1}{2} \begin{bmatrix} 2k_R & c_2 \\
2k_R & \lambda_{min}(J) \end{bmatrix}, W_{12} = \begin{bmatrix} 2k_R \\
\frac{1}{2}c_2 \end{bmatrix} \begin{bmatrix} 1 \\
\frac{1}{2} \lambda_{max}(J) \end{bmatrix}
\]

Our work focuses on the derivative of the Lyapunov function where the UAV dynamics appear. After a considerable amount of manipulations,

\[
V_Q = e_\omega \cdot J e_\omega + k_R e_\omega \cdot e_Q + c_2 e_\omega \cdot e_Q + c_2 e_\omega \cdot e_Q
\]

\[
\leq -z^TW_{13}z + \left( \frac{1}{4k_R} + \frac{\lambda_{max}(J)}{4k_Q \lambda_{min}(J)^2} \right) \|e_Q\|^2
\]

\[
- \frac{c_2^2}{\lambda_{max}(J)} \|e_Q\|^2 - k_c \|e_\omega\|^2 - \frac{k_c}{2} \frac{\|e_Q\|^2}{2k_Q}
\]

\[
W_{13} = \begin{bmatrix} \frac{c_2 k_Q}{\lambda_{min}(J)} & \frac{k_Q c_2}{\lambda_{min}(J)} \\
\frac{k_Q c_2}{\lambda_{min}(J)} & k_Q \end{bmatrix}
\]

Arriving at the inequality,

\[
V_Q \leq -\lambda_{min}(W_{13}) \|z\|^2 + \left( \frac{1}{4k_R} + \frac{\lambda_{max}(J)}{4k_Q \lambda_{min}(J)^2} \right) \|e_Q\|^2
\]

\[
c_2 < \min \left\{ 2k_Q, \sqrt{2k_R \lambda_{max}(J)}, \frac{\lambda_{max}(J)}{\lambda_{max}(J)} \left( \frac{k_Q \lambda_{min}(J)}{2k_Q \lambda_{min}(J)} + \frac{k_Q c_2}{\lambda_{min}(J)} + k_Q \right) \right\}
\]

and $W_{11}, W_{12}, W_{13}$ are positive definite. Furthermore, the following inequality holds for $\lambda_Q = \lambda_{min}(W_{13})/\lambda_{max}(W_{12})$,

\[
V_Q \leq -\lambda_Q V_Q + \phi_Q
\]

\[
\phi_Q = \left( \frac{1}{4k_R} + \frac{\lambda_{max}(J)}{4k_Q \lambda_{min}(J)^2} \right) \|e_Q\|^2
\]

**Boundedness:** Utilizing (24), (27a), if the states are in, $L_n = \{(Q, \dot{Q}, \omega) \in SO(3) \times \mathbb{R}^3 ||\| > \sqrt{\phi_Q/\lambda_{min}(W_{13})}\}$ then $V_Q < 0$. For initial conditions in $L_{\beta} = \{(Q, \dot{Q}, \omega, e_Q, e_\omega) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 V_Q < \lambda_{min}(W_{11})\}$ then $L_n \subseteq L_{\beta}$. Finally for,

\[
\phi_Q < \lambda_{min}(W_{13})/\lambda_{min}(W_{12})
\]

\[
\text{then } L_{n} \subseteq L_{\beta} \text{ and the attitude errors exponentially converge to } L_{n} \text{ and are uniformly ultimately bounded. The superscript (\(c\)) denotes the complement set of (\(\cdot\)). The estimated ultimate bound is,}
\]

\[
\|z\|^2 \leq \lambda_{max}(W_{12})/\lambda_{min}(W_{13}) \phi_Q
\]

Condition (28) is required to ensure that $L_{n} \subseteq L_{\beta}$ and the disturbance term is small enough such that the states remain in $L_n \subseteq L_{\beta}$. $L_{n}$ can be reduced by increasing the $k_c, k_Q$ gains.

**B. Position Stability.** The position error dynamics are calculated by substituting $bF_{p,i} = bF_{p,i} + bF_{p,i} \epsilon_i$ in (3a), where $bF_{p,i}$ is the thrust tracking error, and the resulting expression in (4a) follows by the position component of (8) to get,

\[
m_v \dot{e}_v = m_v e_v - c_x k_c e_x + e_x/m_b
\]

\[
\epsilon_x = Q(A_F bF_{p,1} + bF_{p,2} e_! + bF_{p,3} + bF_D + \sum_{i=1}^{3} (-m_p b_i \epsilon_i))
\]

where $\epsilon_x$ is the position disturbance term that includes VA tracking errors, wind disturbances and VA related accelerations ($\epsilon_i$) see (A2), that due to the small mass/inertia properties of the VA, are dealt as disturbances. The following Lyapunov function is employed, with $c_x > 0$,

\[
V_x = \frac{1}{2} \|e_v\|^2 + \frac{1}{2} c_x k_c e_v \|e_x\|^2 + c_x e_x \cdot e_v
\]

and for $z = [\|e_\omega\|; \|e_Q\|] \in \mathbb{R}^2$ the following inequality holds,

\[
\lambda_{min}(\Pi_{11}) \|z\|^2 \leq V_x \leq \lambda_{max}(\Pi_{12}) \|z\|^2
\]

\[
\Pi_{11} = \frac{1}{2} \begin{bmatrix} c_x k_c k_v & -c_x \\
-c_x & 1 \end{bmatrix}, \Pi_{12} = \frac{1}{2} \begin{bmatrix} c_x k_c k_v & c_x \\
c_x & 1 \end{bmatrix}
\]

where $\lambda_{min, max}(\cdot)$ denotes the min, max eigenvalue of (\(\cdot\)). Differentiating (31), substituting (30), $k_v = k + k_Q$ and $k = k_c + k_Q$ to the resulting equation ($k_x, k_T, k_T, k_Q > 0$, after considerable manipulations, we arrive at the inequality,

\[
V_x \leq -z^T W_1 z + 1/4 (k_x + k_Q) \|e_x\|^2
\]

\[
- \frac{c_2}{\lambda_{min}(W_{13})} \frac{k_Q}{\lambda_{min}(J)} \|e_Q\|^2 - \frac{k_c}{2} \|e_\omega\|^2
\]

\[
\Pi_{13} = \left[ \frac{c_x k_q}{k_T}, \frac{k_T}{\lambda_{min}(J)} \right] \frac{k_T}{(k_T k_x - k_T - k_v)} - \frac{k_T}{4k_T k_x}
\]

\[
\|z\|^2 \leq \lambda_{max}(\Pi_{13}) \|z\|^2
\]

and $\Pi_{11}, \Pi_{12}, \Pi_{13}$ are positive definite. Furthermore, the following inequality holds for $\lambda_x = \lambda_{min}(\Pi_{13})/\lambda_{max}(\Pi_{12})$,

\[
V_x \leq -k_x V_x + \phi_x
\]

\[
\phi_x = \frac{k_x + k_Q}{4k_T k_x k_T} \|e_x\|^2
\]

**Boundedness:** Using (32), (33b), if the system evolves in $L_3 = \{(e_Q, e_\omega) \in \mathbb{R}^3 \times \mathbb{R}^3 ||z|| > \sqrt{\phi_x/\lambda_{min}(\Pi_{13})}\}$ then $V_x < 0$. For initial conditions in $L_3 = \{(Q, \dot{Q}, \omega, e_Q, e_\omega) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 V_x < \lambda_{min}(\Pi_{11}) e_{\omega, max}, V_Q < \lambda_{min}(W_{11})\}$ where $e_\omega(0)$ is $e_{\omega, max}$ $\in \mathbb{R}$ then the solution is also in $L_3 \subseteq L_2$. Finally for,

\[
\phi_x < (\lambda_{min}(\Pi_{11}) \lambda_{min}(\Pi_{13}) e_{\omega, max})/\lambda_{max}(\Pi_{12})
\]

then $L_{\beta} \subseteq L_\beta$ and $e_\omega, e_Q$ exponentially converge to $L_{\beta}$ and are uniformly ultimately bounded. The ultimate bound is,

\[
\|z\|^2 \leq \lambda_{max}(\Pi_{12})/\lambda_{min}(\Pi_{13}) \phi_x
\]

Condition (36) is required to ensure that $L_{\beta} \subseteq L_\beta$ and the disturbance term is small enough such that the states do not exit $L_3$. $L_{\beta}$ can be reduced by increasing the $k_c, k_Q$ gains.

**C. System Position/Attitude Stability.** A Lyapunov function for the complete system is,

\[
\lambda_0 ||z||^2 \leq \{V = V_x + V_Q \} \leq \lambda_M ||z||^2
\]

\[
\lambda_0 = \min \{\lambda_{min}(\Pi_{11}), \lambda_{min}(W_{11})\}
\]

\[
\lambda_M = \max \{\lambda_{max}(\Pi_{12}), \lambda_{max}(W_{12})\}
\]

where $z = [z_c; z]$. Its derivative is given by,

\[
\dot{V} \leq -\lambda_0 V_x + \phi_x + \phi_M = \min \{\lambda_x, \lambda_Q\}
\]

\[
\phi = \phi_x + \phi_M
\]

\[
\phi = \phi_x + \phi_M
\]

\[
(39)
\]

\[
\|z_c(t)\| \leq \sqrt{\lambda_M/\lambda_0} ||z_c(0)|| e^{-\lambda_0 t} + \sqrt{\phi_0/\lambda_M}
\]
**Boundedness:** Using (37), (38), if the system evolves in 
\( L_\eta = \{(Q, \omega, x, v) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 | \|z_c\| > \sqrt{\phi/\lambda_m \lambda_c} \} \) then \( \dot{V} < 0 \). For initial conditions in 
\( L_\rho = \{(Q, \omega, x, v) \in SO(3) \times \mathbb{R}^3 \times \mathbb{R}^3 \times \mathbb{R}^3 | V < \lambda_0 (1 + \epsilon_{\text{max}}) \} \) then the solution is also in \( L_\beta \). Finally, 
\( \phi < \lambda_m \lambda_c (1 + \epsilon_{\text{max}}) \) \hspace{1cm} (41)

then \( L_\eta \subset L_\rho \) and \( \epsilon_q, \epsilon_w, \epsilon_x, \epsilon_v \) exponentially converge to \( L_\eta \) and are uniformly ultimately bounded. The estimated ultimate bound is,

\[ \|z_c\|^2 \leq \phi/\lambda_m \lambda_c \]

Condition (41) ensures that \( L_\eta \subset L_\rho \) and that the disturbance terms are small enough such that the states remain bounded within \( L_\rho \). \( L_\eta \) can be reduced by increasing the \( k_Q, k_w, k_x, k_m \) gains.

**V. SIMULATION RESULTS**

The effectiveness of the proposed strategy is verified through simulations. The system parameters are:

\[ \begin{align*}
\hat{b}_p &= [0.1940; -0.1120; 0][m], \quad \hat{b}_p &= [0.0.2240; 0][m] \\
\hat{b}_p &= [-0.1120; 0][m] \\
b_T &= 3.409 \cdot 10^{-4}[N \cdot (s/rad)^2], \quad b_T &= 7.444 \cdot 10^{-3}[m] \\
d &= 0.014[m], \quad m_b = 1[kg], \quad m_p = 0.060[kg] \\
J &= \text{diag}(0.0205, 0.0211, 0.0344)[kg \cdot m^2] \\
J_p &= \text{diag}(0.1341, 0.1341, 0.0919) \cdot 10^{-4}[kg \cdot m^2]
\end{align*} \]

The controller parameters are chosen to satisfy (26), (34). A preliminary estimate is calculated first through pole placement by choosing desired time constants coefficients:

\[ \begin{align*}
k_R + k_Q &= \text{diag}(2.1938, 2.2607, 3.6840) \in \mathbb{R}^{3 \times 3} \\
k_w + k_Q &= \text{diag}(0.4241, 0.4371, 0.7122) \in \mathbb{R}^{3 \times 3} \\
k_x &= 100, \quad k_x = 20, \quad \gamma = 144, \quad \Delta = 360000, \quad \eta = 1200
\end{align*} \]

A complex flight maneuver for which the UAV firstly recovers from being upside down and then follows a desired pose trajectory will be carried out. The initial conditions are

\[ \begin{align*}
x(0) &= [-1; -1; 0][m], \quad Q(0) = Q_0, \quad (0, 179^\circ, 0), \quad Q_0 = I \\
v(0) &= [0; 0; 0][m], \quad \omega(0) = \omega_i(0) = [0; 0; 0][rad/s], \quad i = 1, 2, 3 \\
Q_i(t) \text{ is given in the Appendix. The trajectory in E}^3 \text{ is that of an } \text{8}, \text{ while altitude wise the UAV performs a 360° roll maneuver. Analytically,} \\
x_a(t) &= sin(0.03t); \quad sin(0.06t); \quad 2[m], \quad Q(t) = Q_0(0, \frac{\pi t}{105}, 0)
\end{align*} \]

To demonstrate the effectiveness of the controller, the simulation includes parameter errors:

\[ \begin{align*}
b_T &= b_T + 0.033b_T, \quad b_T &= b_T + 0.033b_T, \quad d = d + 0.01d \\
m_b &= m_b + 0.01m_b, \quad m_p &= m_p + 0.03m_p, \quad \hat{J}_p = J_p + 0.03 \hat{J}_p
\end{align*} \]

A bounded disturbance wrench \([\mathbf{F}_D, \mathbf{M}_D]\) simulating the wind under "fresh breeze" conditions (17-21(Knots)) is applied according to [11].

Because the desired trajectory begins at \( x_a(0) = [0; 0; 2][m] \) and the UAV is upside down, there is a large initial position and attitude errors, \( \|e_x\| = 2.4595[m] \) and \( \Psi = 1.9825 \text{ (179° in terms of the roll angle), respectively (see Fig. 2(a,b)). The time scale in Fig. 2(a,b) is uneven in order to show both the transient and steady state response. The UAV recovers in 1s, Fig. 2(a,b), and starts to track the figure "8" trajectory, Fig. 2(c), while performing a 360° rotation around the e_1 body fixed axis. Fig. 2(a,b) also includes an enlarged view of the last 50s of the simulation to highlight the results. The tracking performance is shown in both Fig. 2(a,b) maintaining the position error below \( \|e_x\| < 0.02[m] \) and the attitude error below \( \Psi < 1 \times 10^{-4} \) which is less than 0.009[deg] with respect to an equivalent axis angle rotation.

![Figure 2](image_url)

The VAs are operating in cooperative manner since they show small relative attitude difference wrt each other \( \Psi < 0.006 \), see Fig. 2(d), which is less than 0.54[deg] with respect to an equivalent axis angle rotation. This is attributed to the pseudoinverse allocation strategy. The propeller thrusts
shown in Fig. 2(e) verify that the UAV can overcome gravity while in Fig. 2(f) the propeller speeds can be seen. The vectoring gains generate realistic and realizable control torques that can be seen in Fig. 2(g,h,i). The same simulation but without wind was executed to investigate the behavior of the vectoring controller, with the generated second VA control torques shown in Fig. 2(h). This figure shows that the spikes in Fig. 2(g,i) were due to the wind disturbance verifying the claim that the vectoring controller is smooth, without high frequency chattering.

The vectoring of the motors can be implemented using two gimbals, one mounted on the other with orthogonal pivot axes, avoiding limitations of the actuation apparatus such as finite range for the tilting angles. The advantages gained by utilizing geometric control methodologies lie in that we do not have to worry about critical orientations for the VAs or the base that might arise during operation.

Despite modeling inaccuracies of up to 33% for some parameters and disturbances, the system successfully tracks the desired pose trajectory with independent control over its attitude/translational dynamics (holonomic response) verifying that the UAV can hover and maneuver at any attitude without restrictions.

VI. CONCLUSIONS

The modeling and control of a vectoring tricopter UAV were addressed in this article. The UAV is actuated by three thrust motors, each guided by suitable actuators, forming a platform able to independently track any desired attitude and trajectory. The derivation of the equations of motion was followed by the development of a singularity-free control strategy based on geometric feedback linearization that accounts for the inertial effects of the main body, the motors and of the vectoring actuators. A stability proof was provided. Simulations verified the effectiveness of the tricopter configuration and the developed control design, demonstrating a UAV with independent control over its attitude/translational dynamics and a global operational envelope. Future work includes, the development of a prototype and the experimental implementation of the proposed strategy.

REFERENCES


APPENDIX

The attitude error function as given by [8],

$$\Psi(t) = 2 - \sqrt{1 + r^T Q^2 \hat{Q}}$$ (A1)

Angular velocity and acceleration of the CM of the $i^{th}$ VA,

$$^{n}\omega_i = ^{n}\omega_i^p + \hat{Q}^i \hat{\omega}$$

$$b\hat{v}_i = Q^T \hat{v} + b\hat{v}_i$$ (A2)

Vector space isomorphism where $r \in \mathbb{R}^3$,

$$S(r) = [0, -r_3, r_2, r_1, -r_2, r_1, 0]$$

Attitude through Euler-Angles ($c\gamma_i = \cos \gamma_i$, $s\gamma_i = \sin \gamma_i$),

$$Q_i = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c\gamma_i & s\gamma_i \\ 0 & -s\gamma_i & c\gamma_i \end{bmatrix} = \begin{bmatrix} c\gamma_3 & 0 & s\gamma_3 \\ 0 & c\gamma_1 & 0 \\ -s\gamma_1 & c\gamma_1 \end{bmatrix}$$ (A4)

Matrix $M_i \in \mathbb{R}^{15 \times 15}$, from (5), where $M_{i,j} \in \mathbb{R}^{3 \times 3}$,

$$M = \begin{bmatrix} M_{11} & M_{12} & M_{13} & M_{14} & M_{15} \\ M_{21} & M_{22} & M_{23} & M_{24} & M_{25} \\ M_{31} & M_{32} & M_{33} & 0 & 0 \\ M_{41} & M_{42} & 0 & M_{44} & 0 \\ M_{51} & M_{52} & 0 & 0 & M_{55} \end{bmatrix}$$ (A5)

$$M_{1,1} = \{m_b + m_{p_i} \} I, M_{1,2} = -Q \sum_{i=1}^{3} S(b_i p_i + Q \cdot m_i p)$$

$$M_{1,i} = -Q Q_i S(m_i p), i = 3 - 5, M_{2,1} = \sum_{i=1}^{3} S(b_i p_i) m_i Q^T Q$$

$$M_{2,2} = J - \sum_{i=1}^{3} S(b_i p_i) m_i S(b_i p_i + Q \cdot m_i p)$$

$$M_{2,i} = -S(b_i p_i) m_i Q_i S(m_i p), i = 3 - 5$$

$$M_{1,1} = Q_i^T S(Q_i \cdot m_i p) m_i Q_i Q_i^T, i = 3 - 5$$

$$M_{1,2} = Q_i^T S(Q_i \cdot (-m_i p)) m_i Q_i S(m_i p), i = 3 - 5$$

$$M_{1,i} = J_p + Q_i^T S(Q_i \cdot (-m_i p)) m_i Q_i S(m_i p), i = 3 - 5$$