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Determination of Rigid-Body Pose from Imprecise Point Position Measurements

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***Abstract*—The determination of a rigid-body position and orientation from the position of a number of its points is one of the fundamental problems in kinematics. This problem arises in robotics, biomechanics, automatic guided vehicles, real-time control of space structures, etc. Under ideal conditions, it is possible to apply classical methods to find a body's position and orientation. However, in the presence of noise, these methods yield results that are unreliable and inconsistent. Two existing and a new method for determining position and orientation from noisy point coordinate data are presented. The theoretical analysis of the methods combined with an extensive simulation process led to conclusions about their behavior in different situations. The proposed method yields better orientation estimates than the other two methods, yielding reliable results both for absolute and relative position measurements and for low and high noise levels.**

***Index terms*–Body coordinates from point coordinates, rigid body motion, rigid body pose estimation.**

I. INTRODUCTION

In kinematics, the study of rigid body motion generally includes three main aspects, the analysis of displacement, velocity and acceleration. This is a fundamental problem in classical mechanics, which makes a comeback due to the current technological advances. Indeed, sensor systems such as the GPS, motion capture systems, or overhead cameras can supply the coordinates of a number of points that belong to the same body. The challenge is then to process the stream of imprecise coordinates data to determine the position, the orientation and possibly other kinematic variables of a moving rigid body as accurately as possible.

Literature on the theory of spatial transformations and displacements, as well as on determining the object position from three or four points, is available in references [1], [2], and [3]. Complications arise when the coordinate data are redundant and imprecise. To overcome this problem, in [4] a procedure based on the Singular Value Decomposition (SVD) is proposed, which requires the coordinates of three or more non-collinear points and provides a least-squares estimation of rigid body transformation parameters. Other approaches take advantage of modern matrix-oriented software that facilitate SVD, such as Matlab. Two new methods for obtaining object position from imprecise and excess point coordinate data have been presented by Gupta and Chutakanonta and [5]. The first method is called SVD/ QR Decomposition Method because it uses a stable SVD decomposition, followed by a QR-decomposition. The second method is similarly called SVD/ QS Decomposition. An extension of this method for estimating object velocity and acceleration states from given point position, velocity and acceleration data is presented in [6].

A different approach to the problem is presented in [7], where representation issues of rigid body transformations are considered to be dependent on the geometric properties of reflected correspondence vectors into a single coordinate frame. The novel representation of rigid body transformations is based on the constraints about distance, angle and projection measurements.

Later, in [8] a vector method for measuring rigid body motion from marker coordinates was presented, including both finite and infinitesimal displacement analysis. This method takes advantage of the linearity of infinitesimal displacement analysis to formulate the equations of finite displacement as a generalization of Rodrigues' formula when more than three points are used. The approach provides simple, linear, closed form formulas to compute both velocities and finite displacements of a rigid body.

In this paper, three methods for determining rigid body position and orientation from imprecise points coordinate data are implemented and evaluated. The first is an application of theoretical kinematics, where a body-fixed coordinate system is defined and used to track rigid body motion. The second method was proposed by Vertechy and Castelli in [9] and is a sophisticated variant of the first, basic method. The third method proposed in this paper is a novel method that addresses the kinematic problem of locating a body without defining a body-fixed coordinate system. It is shown that this method is computationally efficient and yields better orientation estimates than the other two.

II. RIGID BODY KINEMATICS

Let A and P be two points of a rigid body B, the former being a particular reference point, whereas the latter is an arbitrary point of B, as depicted in Fig. 1. The position vector of point A in the original configuration is $\mathbf{a}$, and the position vector of the same point in the displaced configuration, denoted as A', is $\mathbf{a}'$. Similarly, the position vector of point P in the original configuration is $\mathbf{p}$, while in the displaced configuration B', this point is P', its position vector being $\mathbf{p}'$. Furthermore, $\mathbf{p}'$ is to be determined, while $\mathbf{a}$, $\mathbf{a}'$, and $\mathbf{p}$ are given, along with the rotation matrix $\mathbf{R}$. Then, the following holds, [10]:

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$$\mathbf{p'} = \mathbf{a'} + \mathbf{R}\left(\mathbf{p} - \mathbf{a}\right) \quad (1)$$

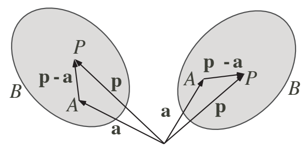


Fig. 1. General rigid body displacement (translation and rotation).

Therefore, knowing the position of a single body point and the body orientation, we can determine the position of any body point. The problem we are interested in this paper is how to best solve the inverse problem, i.e. given a number of rigid body point coordinates, find the coordinates of a reference point and the body orientation. This problem is examined next using a planar example.

We assume that a rigid body in the form of a circular disk of radius *r*, is moving on a plane, see Fig. 2. To determine its motion parameters, position measurements of a number of its points are obtained. With no loss of generality, three non collinear points, P_1, P_2 and P_3, displaced by 120° at distance *r* from the reference point *C*, are selected, see Fig. 2.

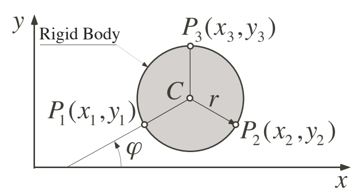


Fig. 2. Rigid body at a plane and three measured points.

Let $\tilde{x}_i$, $\tilde{y}_i$ denote the measured values of the actual x_i, y_i coordinates of point P_i (i = 1,2,3), as given by a position sensor. If the rigid body rotates around *C* by an angle φ, then, the following equations can be deduced:

$$\tilde{x}_i = x_i + n_{x_i} = x_C + r \cdot \cos\left(\theta_i\right) + n_{x_i} \quad (i = 1,2,3) \quad (2)$$

$$\tilde{y}_i = y_i + n_{y_i} = y_C + r \cdot \sin\left(\theta_i\right) + n_{y_i} \quad (i = 1,2,3) \quad (3)$$

where n_{x_i} and n_{y_i} is noise corrupting the measurements and,

$$\theta_i = \varphi + \frac{(i-1)2\pi}{3} \quad (i = 1,2,3) \quad (4)$$

In this case, the problem is to find the angle of rotation φ and the coordinates of *C*, knowing the six measured coordinates of the three points, and their location on the body, described by the radius *r*. It is easy to see that here we have an over-determined system of six equations with three unknowns. Therefore, the solution methodology must manage the data redundancy effectively.

III. Vector Methods

In this paper, vector methods that process imprecise body point position measurements to compute rigid body kinematic parameters are considered or developed.

A. Basic Method

The Basic Method (BM) is an application of fundamental kinematic theory. A body-fixed coordinate frame S_1: $\{\mathbf{l}, \mathbf{m}, \mathbf{n}\}$ is defined and tracks the rigid body motion, see Fig. 3. By tracking the angles of this frame, the orientation of the rigid body with respect to the coordinate system S_o is computed. Note that in the BM, the origin of S_1 is placed at some measured point P_i, see Fig. 3(b).

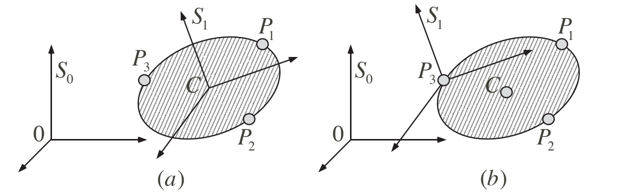


Fig. 3. Rigid body with reference point C and a body-fixed frame S_1. (a) Frame at C, (b) Frame at P_i.

The position vector $\mathbf{c}$ of the reference point is expressed as a weighed sum of vectors $\mathbf{p}_i$ of a number of body points:

$$\mathbf{c} = \sum_{i=1}^{n} w_i \mathbf{p}_\mathbf{i} \quad (5)$$

The coefficients w_i must satisfy,

$$\sum_{i=1}^{n} w_i = 1 \quad (6)$$

and are unique for a given rigid body geometry. For the planar example presented earlier, n = 3 and $w_i = 1/3$.

The body attitude can be determined by the rotation matrix $\mathbf{R}$, as described by the unit vectors in (7), or the x–y–z Euler angles φ, θ and ψ in (8),

$$\mathbf{R} = \left[\mathbf{l} \ \ \mathbf{m} \ \ \mathbf{n}\right] \quad (7)$$

$$\mathbf{R} = \begin{bmatrix} c\theta c\psi & -c\varphi s\psi + s\varphi s\theta c\psi & s\varphi s\psi + c\varphi s\theta c\psi \\ c\theta s\psi & c\varphi c\psi + s\varphi s\theta s\psi & -s\varphi c\psi + c\varphi s\theta s\psi \\ -s\theta & s\varphi c\theta & c\varphi c\theta \end{bmatrix} \quad (8)$$

Using measurements from three points, the vectors $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ can be computed by (9)-(11), as

$$\mathbf{l} = \frac{\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2}{\left|\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_2\right|} = \begin{bmatrix} l_x & l_y & l_z \end{bmatrix}^\mathrm{T} \quad (9)$$

$$\mathbf{m} = \frac{\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right) - \mathbf{l} \cdot \left[\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right) \cdot \mathbf{l}\right]}{\left|\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right) - \mathbf{l} \cdot \left[\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right) \cdot \mathbf{l}\right]\right|} = \left[m_x, m_y, m_z\right]^T \quad (10)$$

$$\mathbf{n} = \mathbf{l} \times \mathbf{m} \quad (11)$$

For the planar problem, where the rigid body rotates around the z axis, the position of point *C* and angle of rotation are given by (12) and (13), respectively:

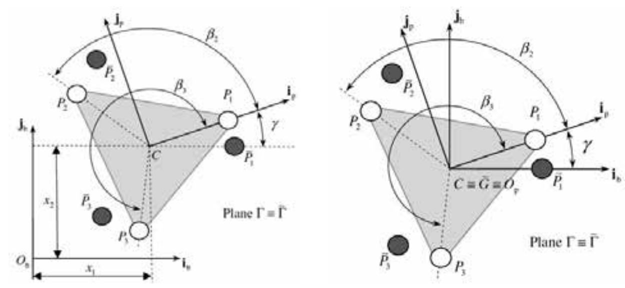
$$\mathbf{c} = \frac{\left(\tilde{\mathbf{p}}_1 + \tilde{\mathbf{p}}_2 + \tilde{\mathbf{p}}_3\right)}{3} = \left(\frac{\tilde{x}_1 + \tilde{x}_2 + \tilde{x}_3}{3}, \frac{\tilde{y}_1 + \tilde{y}_2 + \tilde{y}_3}{3}, 0\right) \quad (12)$$

$$\varphi = \operatorname{atan2}\left(m_x - l_y, m_y + l_x\right) \quad (13)$$

B. Vertechy-Castelli Method

The method proposed by Vertechy and Castelli (VCM) is presented in [9]. The VCM estimates the pose of a rigid body using the locations of three of its points measured by noisy sensors. The position vector is determined by means of (5), where n is three and the w_i (i=1,..,n) are equal to 1/3.

Regarding the orientation, the origin of the body coordinate system S_p is point C, rather than P_3, as in the BM. The orientation problem is set up as a minimization one. From a physical point of view, this minimization corresponds to the solution of the planar static equilibrium problem of a triangle hinged in $\tilde{G}$ and whose vertices P_i are connected to measured points $\tilde{P}_i$ by means of linear springs of constant stiffness, see Fig. 4(b).



(a) (b)

Fig. 4. (a) Measured ($\tilde{P}$) and nominal (P_i) points, (b) Optimal location of reference system S_b, [9].

In this method, the rotation matrix is given by (7) with $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ defined using unit vectors $\mathbf{i}_b$ and $\mathbf{j}_b$, see Fig. 4(b),

$$\mathbf{i}_b = \frac{(\tilde{\mathbf{p}}_1 - \mathbf{c})}{\left|(\tilde{\mathbf{p}}_1 - \mathbf{c})\right|} \tag{14}$$

$$\mathbf{j}_b = \frac{(\tilde{\mathbf{p}}_2 - \mathbf{c}) - \mathbf{i}_b\left((\tilde{\mathbf{p}}_2 - \mathbf{c})\cdot \mathbf{i}_b\right)}{\left\|(\tilde{\mathbf{p}}_2 - \mathbf{c}) - \mathbf{i}_b\left((\tilde{\mathbf{p}}_2 - \mathbf{c})\cdot \mathbf{i}_b\right)\right\|} \tag{15}$$

Then, the coefficients a and b are computed as,

$$a = 2\left[L_2\left(\tilde{q}_{2,1}s_2 - \tilde{q}_{2,2}c_2\right) + L_3\left(\tilde{q}_{3,1}s_3 - \tilde{q}_{3,2}c_3\right)\right] \tag{16}$$

$$b = 2\left[L_1\tilde{q}_{1,1} + L_2\left(\tilde{q}_{2,1}c_2 + \tilde{q}_{2,2}s_2\right) + L_3\left(\tilde{q}_{3,1}c_3 + \tilde{q}_{3,2}s_3\right)\right] \tag{17}$$

where $s_i = \sin(\beta_i)$, $c_i = \cos(\beta_i)$, and

$$\begin{aligned} L_i &= \left|\tilde{\mathbf{p}}_i - \mathbf{c}\right| \quad i = 1,2,3 \\ \tilde{q}_{i,1} &= \left(\tilde{\mathbf{p}}_i - \mathbf{c}\right)\cdot \mathbf{i}_b \\ \tilde{q}_{i,2} &= \left(\tilde{\mathbf{p}}_i - \mathbf{c}\right)\cdot \mathbf{j}_b \end{aligned} \tag{18}$$

and β_i is the angle between vectors $\left(\tilde{\mathbf{p}}_i - \mathbf{c}\right)$ and $\left(\tilde{\mathbf{p}}_1 - \mathbf{c}\right)$. The orientation problem solution is then given by (19)-(22):

$$\hat{\gamma} = -\tan^{-1}\left(\frac{a}{b}\right) \tag{19}$$

where $\hat{\gamma}$ is the angle between $\mathbf{i}_p$ and $\mathbf{i}_b$, shown in Fig. 4(b). Finally, vectors $\mathbf{l}$, $\mathbf{m}$ and $\mathbf{n}$ are given by (20)-(22):

$$\mathbf{l} = \mathbf{i}_b \cos\left(\hat{\gamma}\right) + \mathbf{j}_b \sin\left(\hat{\gamma}\right) \tag{20}$$

$$\mathbf{m} = \mathbf{j}_b \cos\left(\hat{\gamma}\right) - \mathbf{i}_b \sin\left(\hat{\gamma}\right) \tag{21}$$

$$\mathbf{n} = \mathbf{l} \times \mathbf{m} \tag{22}$$

For the planar problem, the position of point C is given again by (12), while the angle of rotation by (13), where l_x, l_y, m_x and m_y are defined by (20) and (21).

IV. Proposed New Method

A. New Method for 3D Problems

In the two previous methods, a body-fixed coordinate frame was defined. Sensory systems introduce measurement noise, apart from the intrinsic computational noise. Consequently, the definition of a body coordinate system is imprecise and amplifies the process imprecision. Thus, a key factor for an accurate and effective solution should be by-passing the determination of a body-fixed coordinate frame.

The proposed New Method for 3D problems (NM-3D) requires measuring the coordinates of four points, which are the vertices of a tetrahedron, see Fig. 5. The NM suggests finding the orientation matrix by determining the three normal vectors at three sides of the tetrahedron formed.

In more detail, let $\mathbf{n}_1$ denote the vector normal to side $P_2P_3P_4$. Then, $\mathbf{n}_1$ is given by,

$$\mathbf{n}_1 = \frac{\left(\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_4\right)\times\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right)}{\left\|\left(\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_4\right)\times\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_3\right)\right\|} \tag{23}$$

Similarly, $\mathbf{n_2}$ and $\mathbf{n_3}$, are given by,

$$\mathbf{n}_2 = \frac{\left(\tilde{\mathbf{p}}_4 - \tilde{\mathbf{p}}_3\right)\times\left(\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_4\right)}{\left\|\left(\tilde{\mathbf{p}}_4 - \tilde{\mathbf{p}}_3\right)\times\left(\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_4\right)\right\|} \tag{24}$$

$$\mathbf{n}_3 = \frac{\left(\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_4\right)\times\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1\right)}{\left\|\left(\tilde{\mathbf{p}}_1 - \tilde{\mathbf{p}}_4\right)\times\left(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1\right)\right\|} \tag{25}$$

If the tetrahedron rotates and $\mathbf{n}_i$ and $\mathbf{n_i}'$ (i=1,2,3) are the normal vectors before and after the rotation respectively, then (26) applies for every $\mathbf{n_i}$.

$$\begin{aligned} &\mathbf{n}_i' = \mathbf{R}\cdot\mathbf{n}_i \Rightarrow \\ &\mathbf{n}' = \begin{bmatrix} \mathbf{n}_1', \ \mathbf{n}_2', \ \mathbf{n}_3' \end{bmatrix}_{3\times 3} = \mathbf{R}\cdot\begin{bmatrix} \mathbf{n}_1, \ \mathbf{n}_2, \ \mathbf{n}_3 \end{bmatrix}_{3\times 3} = \mathbf{R}\cdot\mathbf{n} \end{aligned} \tag{26}$$

And hence,

$$\mathbf{R} = \mathbf{n}'\cdot\mathbf{n}^{-1} \tag{27}$$

Consequently, for 3D motion, the orientation of the rigid body is determined by (8) and (27).

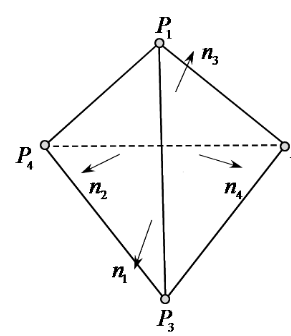


Fig. 5. 3D rigid body motion with four measured points.

B. New Method Variation for 2D Problems

The proposed method for 2D problems (NM-2D) uses a number of coplanar points that define a polygon. Then, the orientation can be found by computing the angle between the normal vector on a polygon' s side before and after the rotation, or, equivalently, the angle between the initial and the rotated side. Better than this, the rotation angle can be obtained as the average of all the calculated angles between the original polygon side and the rotated one.

For example, if three points are tracked forming a triangle, see Fig. 6, and P_i and $P_i^{'}$ (*i=1,2,3*) are points at two consecutive time instants, the rotation angle is given by (28), as follows,

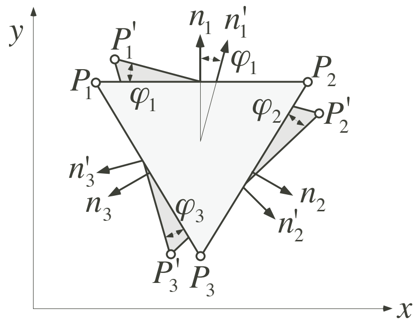


Fig. 6. Two consecutive time instants for the case of three points position measurements during a rigid body motion.

$$\varphi = \frac{\varphi_1 + \varphi_2 + \varphi_3}{3} \tag{28}$$

where

$$\varphi_1 = \cos^{-1}\frac{(\tilde{\mathbf{p}}_2' - \tilde{\mathbf{p}}_1')\cdot(\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1)}{\left\|\tilde{\mathbf{p}}_2' - \tilde{\mathbf{p}}_1'\right\|\left\|\tilde{\mathbf{p}}_2 - \tilde{\mathbf{p}}_1\right\|} \tag{29}$$

$$\varphi_2 = \cos^{-1}\frac{(\tilde{\mathbf{p}}_3' - \tilde{\mathbf{p}}_1')\cdot(\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_1)}{\left\|\tilde{\mathbf{p}}_3' - \tilde{\mathbf{p}}_1'\right\|\left\|\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_1\right\|} \tag{30}$$

$$\varphi_3 = \cos^{-1}\frac{(\tilde{\mathbf{p}}_3' - \tilde{\mathbf{p}}_2')\cdot(\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_2)}{\left\|\tilde{\mathbf{p}}_3' - \tilde{\mathbf{p}}_2'\right\|\left\|\tilde{\mathbf{p}}_3 - \tilde{\mathbf{p}}_2\right\|} \tag{31}$$

In summary, if three points are used, the position of point *C* is given again by (12), while the angle of rotation by (28).

V. Theoretical Comparison of Methods

It is easy to see that the computational complexity of the three methods is the same (computational time increases linearly with the number of operations). Therefore, in order to reach a verdict for the computational requirements, the arithmetic operations should be enumerated. Table 1 presents in detail the number of operations per method and the time requirements of each with respect to time constant c (equal to 1.3998 μs). It is evident that the VCM method has substantially more requirements than the BM. The NM for 3D problems demands less time than the VCM and more than the BM. It is particularly important that its variation for 2D problems demands almost 63% less time than that of the VCM. This attribute is of particular importance, since a large number of operations except the large computational delay, also introduce increased numerical noise into the process.

Table I

Time Requirements per Method

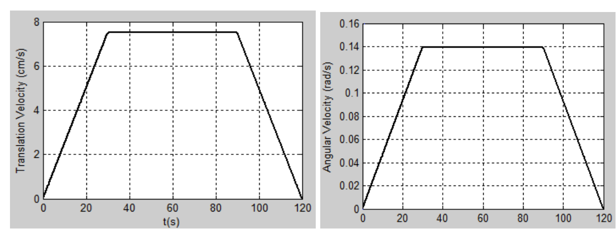
Operation	Duration	Number of Operations			
		BM	**VCM**	**NM-3D**	**NM-2D**
Add/sub, Mul/div	1c	12	54	9	9
Dot product	5c	1	9	-	3
Cross product	9c	1	1	3	-
Vector Norm	5c	2	8	3	6
3x3 Matrix Mul.	15c	-	-	1	-
3x3 Matrix Inv.	42c	-	-	1	-
Trigonometric	2c	1	16	-	3
Root Extraction	2c	2	8	3	6
Total Time		**41c**	**196c**	**114c**	**72c**

VI. Simulation Results

A. Experimental Setup and Parameters

Extensive simulations took place in order to reach reliable conclusions about the behavior of each method in different environments. The methods were implemented in code executable by Matlab.

It is assumed that a rigid body performs a complex motion, translating and rotating in 2D for 120 s. The translation and angular velocity trajectories are depicted in Fig. 7(a) and (b), respectively. The maximum translational velocity is 7.5 cm/s and the maximum angular velocity is 0.14 rad/s. In the interval of 120 s, the rigid body travels a distance of 6.75 m and performs two rotations around C. Fig. 8 depicts the rigid body in its initial and final position, as well as the successive positions of points P_1, P_2 and P_3.



(a) (b)

Fig. 7. Body velocity trajectories of (a) translation, and (b) rotation.

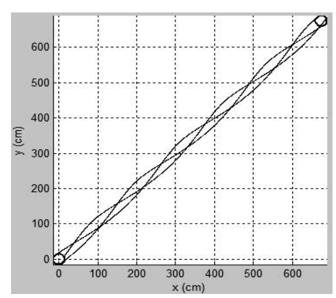


Fig. 8. Initial and final position of the rigid body. Also shown are the successive positions of the three tracked points.

The input data for all methods is the Cartesian coordinates of three non-collinear points of the body. Here, three points were used, placed around a circle of diameter 0.30 m. The exact values of the coordinates of three points were corrupted by adding white noise to simulate the actual measurements, as reflected by a relative or absolute position sensor. Specifically, let x_{act} denote the actual value of the x

coordinate. The corresponding measured value $\tilde{x}$ for absolute position sensor is given by,

$$\tilde{x} = x_{act} + N(0, \sigma) \tag{32}$$

i.e. the actual measurement is corrupted by while noise of zero mean and with covariance σ. For the relative position sensor, the measured value is given by,

$$\delta\tilde{x} = \delta x_{act} + N\left(0,\, \delta x_{act} \frac{a}{100}\right) \tag{33}$$

i.e. the actual incremental displacements are deteriorated by additive white noise with zero mean and covariance equal to $a\%$ times the actual value. This model was obtained by experiments with sensors used in computer mice. The measured y values were corrupted similarly.

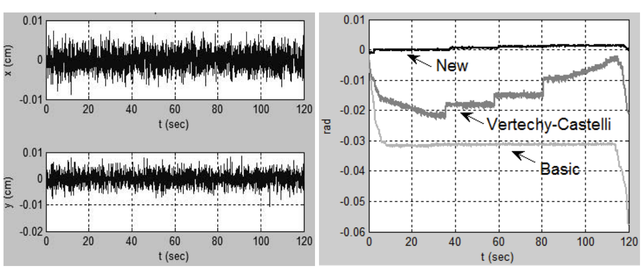
The output of all methods is the rotation angle of the body and the position of a particular body point, namely of point C. The error is considered to be the absolute difference between the actual and the estimated value.

B. Results

In this section, we present simulation results that demonstrate the performance of the three methods for 2D problems. The results are depicted by a graphical representation of errors in position and orientation, as well as, by the statistical analysis of the orientation errors. Regarding the graphical representation of the position errors, only one plot is shown as the method for determining position errors is common for all three methods. The tables that follow every plot contain the statistical analysis of the orientation error per method.

1) Absolute Position Sensor

Fig. 9 shows the position and orientation errors for the case of white noise with mean zero and covariance σ = 0.00417 cm. It is evident that the VCM performs better than the BM. The performance of the NM is by far the best one, as evidenced by Fig. 9(b) and Table II.



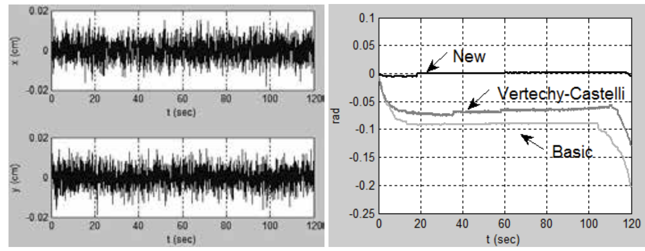
(a) (b)

Fig. 9. 3σ = 0.0125 cm. (a) Position error, (b) Orientation error.

TABLE II
STATISTICAL ANALYSIS OF ORIENTATION ERROR (3σ=0.0125cm)

	BM	VCM	NM
Mean	-0.0311	-0.0141	$7.3223*10^{-4}$
Range	0.0577	0.0232	0.0031
Variance	$1.8615*10^{-5}$	$3.212*10^{-4}$	$3.8226*10^{-7}$
Standard Deviation	0.0043	0.0179	$6.1827*10^{-4}$
Coeff. of Variation	-0.1385	-1.2741	0.8444

Fig. 10 shows the position and orientation errors for the case of white noise with mean zero and covariance σ = 0.00833 cm. It is evident that results of VCM are slightly better than that of BM. However, the performance of the NM is the best one, again as evidenced by Fig. 10(b) and Table III.



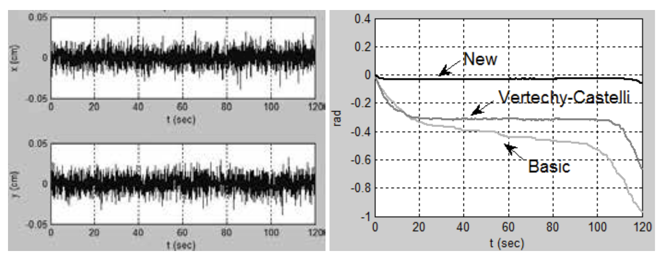
(a) (b)

Fig. 10. 3σ = 0.025 cm. (a) Position error, (b) Orientation error.

TABLE III
STATISTICAL ANALYSIS OF ORIENTATION ERROR (3σ=0.025cm)

	BM	VCM	NM
Mean	-0.0924	-0.0668	$1.1409*10^{-4}$
Range	0.2027	0.1303	0.0096
Variance	$4.846*10^{-4}$	$7.8374*10^{-4}$	$5.9463*10^{-4}$
Standard Deviation	0.022	0.028	0.0024
Coeff. of Variation	-0.2383	-0.419	21.3735

Fig. 11 shows the position and orientation errors for the case of white noise with mean zero and covariance σ = 0.01667 cm. The results of VCM are better than that of BM, while the performance of the NM is again by far the best one, evidenced by Fig. 11(b) and Table IV.



(a) (b)

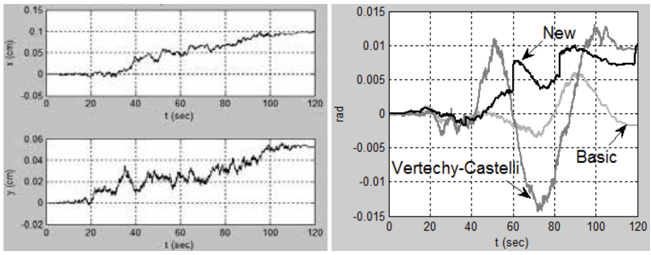
Fig. 11. 3σ = 0.05 cm. (a) Position error, (b) Orientation error.

TABLE IV
STATISTICAL ANALYSIS OF ORIENTATION ERROR (3σ=0.05cm)

	BM	VCM	NM
Mean	-0.4387	-0.3137	-0.0304
Range	0.9687	0.6697	0.0596
Variance	0.0287	0.0219	$2.2332*10^{-5}$
Standard Deviation	0.1695	0.1479	0.0047
Coeff. of Variation	-0.3864	-0.4714	-0.155

2) Relative Position Sensor

Here, the measurements are taken by a relative position sensor, i.e. only incremental displacements are available, as opposed to absolute positions. Fig. 12 shows the position and orientation errors for the case of white noise with mean zero and a = 1%. For this extremely low noise level the BM performs better than both the VCM and the NM, see also Table V.



(a) (b)

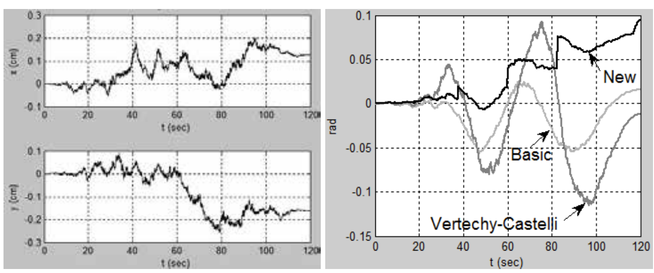
Fig. 12. Noise 1%. (a) Position error, (b) Orientation error.

TABLE V

STATISTICAL ANALYSIS OF ORIENTATION ERROR (noise a=1%)

	BM	VCM	NM
Mean	$6.002*10^{-5}$	0.0013	0.0038
Range	0.0095	0.0274	0.0118
Variance	$4.5484*10^{-6}$	$5.2488*10^{-5}$	$1.4142*10^{-5}$
Standard Deviation	0.0021	0.0072	0.0038
Coeff. of Variation	35.533	5.7315	0.9854

Fig. 13 shows the position and orientation errors for the case of white noise with mean zero and a = 5%. With this noise level, the BM performs better than the VCM and slightly better than the NM, see Fig. 13(b) and Table VI.



(a) (b)

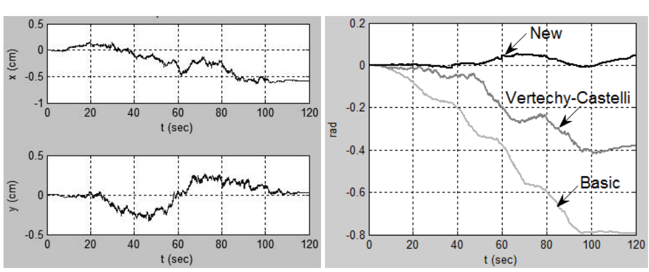
Fig. 13. Noise 5%. (a) Position error, (b) Orientation error.

TABLE VI

STATISTICAL ANALYSIS OF ORIENTATION ERROR (noise a=5%)

	BM	VCM	NM
Mean	-0.0124	-0.0158	0.0325
Range	0.0815	0.2092	0.102
Variance	$5.0453*10^{-4}$	0.0026	$9.1753*10^{-4}$
Standard Deviation	0.0225	0.0508	0.0303
Coeff. of Variation	-1.8171	-3.2245	0.9317

Fig. 14 shows the position and orientation errors for the case of white noise with mean zero and a = 10%. With that noise level, BM is almost inapplicable, while results of the VCM method are better. However, the NM performs better than both, as shown in Fig. 14(b) and Table VII.



(a) (b)

Fig. 14. Noise 10%. (a) Position error, (b) Orientation error.

TABLE VII

STATISTICAL ANALYSIS OF ORIENTATION ERROR (noise a=10%)

	BM	VCM	NM
Mean	-0.4186	-0.187	0.0149
Range	0.7953	0.4184	0.0691
Variance	0.081	0.0769	$3.4958*10^{-4}$
Standard Deviation	0.2847	0.2773	0.0187
Coeff. of Variation	-0.6801	-1.4825	1.2568

VII. CONCLUSIONS

The problem of rigid body position determination from imprecise position measurements of a number of its points was studied. To determine rigid body motion parameters, namely position and orientation, three methods for 3D and one for 2D problems were presented. Regarding 3D problems, the first method is based on theoretical kinematics and is referred to as the BM, while the second one was proposed by Vertechy and Castelli (VMC). The third method is a novel one proposed here and referred to as the NM. The NM computes the orientation using a number of normal vectors. The fourth method is a variation of the NM for 2D problems and is based on a very simple idea, which takes advantage of the data redundancy, providing reliable results.

A planar kinematic problem was studied in which imprecise coordinate data for three points were used in determining position and orientation. The BM, the VCM, and the NM-2D were compared. It was found that when an absolute sensor was used, the NM-2D outperforms both other methods. With a relative position sensor, the BM gives slightly better results when the noise level is lower than 5%, and is comparable to the results obtained by the NM-2D. In cases where the noise is more than 5%, the BM and the VCM are inapplicable, while the NM performs efficiently. Therefore, in general the NM, both with the absolute and the relative position sensor and for low and high noise level, yields reliable results. This is probably due to the fact that the arithmetic operations in the NM are less, and only absolute differences between noisy measurements are used.

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