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An optimal design approach for tracking problems and its assessment against classical controllers

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A class of linear regulator problems, referred to as tracking problems, impose the requirements of zero steady state error, elimination of disturbance effects and immunity to system parameter changes. A controller design approach for this problem, using linear optimal control theory, is presented, and its advantages are outlined by comparison with previous modern and classical approaches.

1. Introduction

In practical control systems design, it is frequently required to keep a controlled variable at a constant value, over a long period of time, in the presence of disturbances or loads. Of course, this constant value—the set point—may change from time to time (tracking problem).

In the present study we are concerned with a class of control problems, neglected theoretically and useful practically, in which the control system should satisfy the following conditions :

(*a*) the steady state error of a desired reference input should be zero ;
(*b*) the effect of constant but unknown disturbances should be completely removed from the steady state output ; and
(*c*) the output should not be susceptible to system parameter changes, while conditions (*a*) and (*b*) continue to be valid.

In general, there are two basic approaches to the controller design for the regulator problem. One is to follow the classical design methods, the other to apply the optimal linear regulator theory. The object of this study is to propose a design approach to the aforementioned problem and give a performance assessment of optimal controllers as well as classical pseudo derivative feedback (PDF) and PID controllers.

Firstly, the structure of the PDF controller is considered, and then some of the issues concerning the effectiveness of the optimal controllers are discussed. Secondly, the optimal controller for the regulator problem is derived, using the ‘ error ’ coordinates system representation. An assessment of performance and a comparison between the PDF and optimal regulator is obtained for a simple system. Subsequently, a new design procedure is developed and a performance comparison of the relevant controllers is presented. Finally, some interesting aspects for controlling higher order systems are briefly discussed.

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2. The PDF controller

Phelan (1977) has discussed different control schemes, and ultimately suggested that the PDF controller constitutes an optimum scheme for all types of plants.

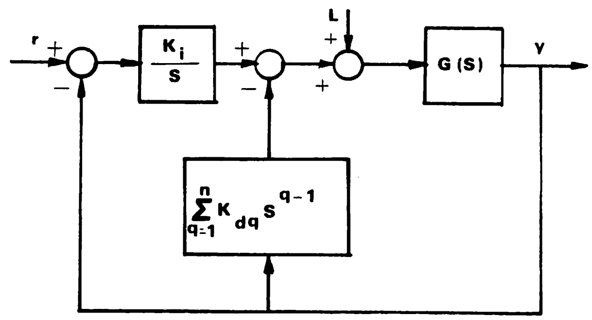


Figure 1. Block diagram of PDF controller for a nth-order plant.

Figure 1 depicts the block diagram of an nth-order plant with PDF control. It is seen that, PDF control is realized by feeding back the output y and its derivatives up to $(n-1)$ order, as well as by introducing integral control action.

For an nth-order plant with the following transfer function

$$G(s)=\frac{1}{s^n+a_{n-1}s^{n-1}+\ldots+a_1s+a_0} \tag{1}$$

the resulting operational equation, relating the output y to the input r and the load L is given by

$$y=\frac{K_i}{s^{n+1}+(K_{d_n}+a_{n-1})s^n+\ldots+(K_{d_1}+a_0)s+K_i}r + \frac{s}{s^{n+1}+(K_{d_n}+a_{n-1})s^n+\ldots+(K_{d_1}+a_0)s+K_i}L \tag{2}$$

It is clear from eqn. (2), that, by adjusting the parameters K_{dj} $(j=1, \ldots, n)$ and K_i, it is possible to place the poles of the controlled system at any desired point. This fact is also a characteristic of the state feedback optimal controller.

The properties and some serious limitations of the PDF control scheme will be discussed in §§ 5 and 6.

3. Approaches to the optimal regulator design

Consider the nth-order, single-input–single-output linear time invariant system

$$\dot{\mathbf{x}}=A\mathbf{x}+\mathbf{b}u \tag{3 a}$$

$$y=\mathbf{c}^T\mathbf{x} \tag{3 b}$$

Let the optimal control law be the one that minimizes a quadratic performance index of the form

$$J = \int_0^\infty (\mathbf{x}^T Q \mathbf{x} + u^2)\, dt \tag{4}$$

For the case of a non-zero set point, it can be shown (Kwakernaak and Sivan 1972) that the optimal control law is

$$u(\mathbf{x}) = -\mathbf{K}^T \mathbf{x} + G_c^{-1}(0) r \tag{5}$$

where $\mathbf{K}^T$ is the feedback coefficient vector and $G_c(0)$ the closed-loop transfer function at $s=0$.

A drawback of the control law (5) is, that it drives the steady state error e_{ss} to zero, only in the case of constant and accurately known system parameters. Moreover, this control law cannot eliminate the influence of disturbances or loads from the output variable. Thus, the optimal controller given by eqn. (5) does not fulfil the requirements stated earlier, and therefore cannot yield an acceptable design solution to the regulator problem.

A possible method of approach to this problem, proposed by Johnson (1968, 1970), is as follows : for $t \rightarrow \infty$, the state $\mathbf{x} \rightarrow \mathbf{x}_{ss}$, the output $y \rightarrow r$ and from eqn. (5) the control input tends to a constant, whose value is not known *a priori*, i.e.

$$\lim_{t \rightarrow \infty} u(t) = c \tag{6}$$

so he introduces the equivalent relation

$$\lim_{\rightarrow \infty} \dot{u}(t) = 0 \tag{7}$$

The last condition is introduced into an alternate formulation of the optimal control problem, by penalizing $\dot{u}$, instead of u, in the cost functional. This approach will not be considered further, first, because it is applicable only to systems in a phase variable representation, with $y = x_1$, having no open loop zeros, and second, since for high order systems, the approach yields impractical controllers, involving high order derivatives.

Perhaps, the most interesting offspring emanating from this approach is the need of an integrator in the forward path to keep the steady state error to zero. This, for a first order system, yields the well known PI controller (Johnson 1968, 1970, Athans 1971).

Another approach is to formulate the problem in ' error ' coordinate form and then introduce an integrator in the forward path. It is summarized below.

4. Optimal integral control using error coordinates

A different formulation of the optimal regulator problem is based on the error coordinates representation (Hedrick 1978). For an nth-order single-input–single-output system, a new state vector $\tilde{\mathbf{x}}$ is formed, containing the error $e = y - r$ and an $(n-1)$th-order vector $\mathbf{x}_2$ such that

$$\mathbf{x}_2 = T_1 \mathbf{x} \tag{8}$$

where T_1 is an $(n-1)\times n$ matrix. Letting $y=\mathbf{c}^T\mathbf{x}$, the new state equation becomes

$$\dot{\tilde{\mathbf{x}}}=[\mathbf{c} \mid T_1{}^T]^T\dot{\mathbf{x}}=T_2\dot{\mathbf{x}} \tag{9}$$

$$\tilde{\mathbf{x}}=T_2\mathbf{x}-[1\ \ 0\ \ \dots\ \ 0]r \tag{10}$$

The matrix T_1 must be chosen so that T_2 is non-singular.

Supposing that the system is subjected to a load $\mathbf{L}$, eqn. (3 *a*) becomes

$$\dot{\mathbf{x}}=A\mathbf{x}+\mathbf{b}u+\mathbf{L} \tag{11}$$

By combining eqn. (11) with (9) and (10) yields

$$\dot{\tilde{\mathbf{x}}}=\tilde{A}\tilde{\mathbf{x}}+\tilde{\mathbf{b}}u+\tilde{\mathbf{L}} \tag{12}$$

where

$$\tilde{A}=T_2AT_2{}^{-1} \tag{13 a}$$

$$\tilde{\mathbf{b}}=T_2\mathbf{b} \tag{13 b}$$

$$\tilde{\mathbf{L}}=T_2L+T_2AT_2{}^{-1}[1\ \ 0\ \ \dots\ \ 0]r \tag{13 c}$$

To achieve zero steady state error, $e_{ss}=0$, an integrator is introduced, whose action is taken into account by the equation

$$\dot{\tilde{q}}=e=y-r \tag{14}$$

The following augmented state vector can now be constructed

$$\boldsymbol{\xi}=\begin{bmatrix}\tilde{\mathbf{x}}\\ \\ \tilde{q}\end{bmatrix} \tag{15}$$

and the augmented system equation is

$$\dot{\boldsymbol{\xi}}=\bar{A}\boldsymbol{\xi}+\bar{\mathbf{b}}u+\bar{\mathbf{L}} \tag{16}$$

where $\bar{A}$, $\bar{\mathbf{b}}$ and $\bar{\mathbf{L}}$ are easily found.

The control law for this problem is in the form

$$u=-K_e e-K_{x_2}x_2-K_{\tilde{q}}\int_0^t e\,dt \tag{17}$$

The approach is illustrated by the following example, which will be retained for comparison in the following sections.

Consider the linear plant

$$\dot{x}=-\frac{1}{a}u \tag{18 a}$$

$$y=x \tag{18 b}$$

Here, $n=1$, $T_1=0$, $T_2=1$, $\tilde{\mathbf{x}}=e_1=e=y-r$. Defining $\tilde{q}=e_2$, eqn. (18 *a*) becomes

$$\dot{e}_1=-\frac{1}{a}u \tag{19 a}$$

$$\dot{e}_2=e_1 \tag{19 b}$$

Let the cost functional to be minimized

$$J=\int_0^\infty (\rho e_1{}^2+qe_2{}^2+u^2)\,dt \tag{20}$$

The solution of the associated Riccati equation gives the feedback coefficients K_1 and K_2 for the optimal control law

$$u = K_1 e_1 + K_2 e_2 = K_1 e_1 + K_2 \int_0^t e\,dt \qquad (21\,a)$$

where

$$K_1 = (2a\sqrt{q} + \rho)^{1/2} \qquad (21\,b)$$

$$K_2 = \sqrt{q} \qquad (21\,c)$$

The control law (21 *a*) reflects the design of a PI controller with optimal parameter values.

5. PDF compared with optimal control

The closed loop operational equation, for the plant of eqn. (18), subjected to a load, L, with PDF control is

$$y = \frac{K_i}{as^2 + K_{d_1}s + K_i}\,r + \frac{s}{as^2 + K_{d_1}s + K_i}\,L \qquad (22)$$

On the other hand, the optimal control law of eqn. (21 *a*) leads to the following operational equation

$$y = \frac{K_1 s + K_2}{as^2 + K_1 s + K_2}\,r + \frac{s}{as^2 + K_1 s + K_2}\,L \qquad (23)$$

Equations (22) and (23) are identical except for a closed loop zero introduced by law (21 *a*). The influence of this zero will be investigated according to certain classical performance criteria.

The closed loop system is of second-order, therefore, as is well known, the overshoot $M_p\%$ and the product $\omega_n t_p$ (natural frequency times the first peak time) are functions of the damping ratio ζ. Figures 2 and 3 depict these relations for the cases of PDF and optimal control.

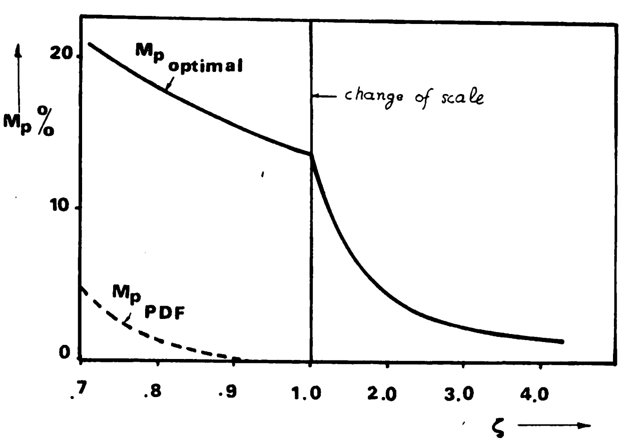


Figure 2. Per cent overshoot M_p of system (18) with PDF and optimal control.

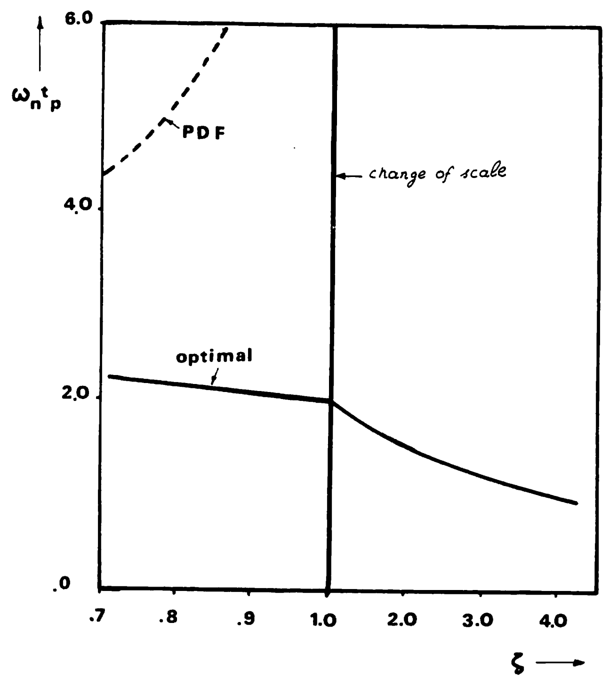


Figure 3. The quantity $\omega_n t_p$ as a function of ζ for system (18) with PDF and optimal control.

It is clear, that the optimal controller induces a faster response with larger overshoots. Of course, the results of Figs. 2 and 3 provide a partial picture of the effects of the two controllers. However, they deem to be useful, since they relate performance characteristics, (as M_p and t_p), to closed-loop pole location (implicitly via ω_n and ζ).

The performances of the two systems are now compared for various design constraints. The numerical values adopted are $a=3$ and $r=10$.

Case 1. *Equal* $u_{\max}/r$ *and* ζ $(L=0)$

Here, the maximum value of the control u and the damping ratio ζ are the same for the two systems. Let $u_{\max}=5$ and $\zeta=0{\cdot}707$.

Through simulation, the resulting PDF parameter values to meet the above conditions are

$$K_{d_1}=1{\cdot}548, \quad K_i=0{\cdot}399$$

or

$$\omega_n=(K_i/a)^{1/2}=0{\cdot}365, \quad \zeta=K_{d_1}/2(aK_i)^{1/2}=0{\cdot}707$$

For the optimal control of eqns. (21)–(23), since $\zeta=0{\cdot}707$ ρ has to be zero. Then, by simulation it is found

$$q=0{\cdot}00174, \quad \rho=0$$

or

$$\omega_n=(\sqrt{(q)}/a)^{1/2}=0{\cdot}118, \quad \zeta=(2a\sqrt{(q)}+\rho)^{1/2}/(4a\sqrt{q})^{1/2}=0{\cdot}707$$

The response to a step input $r = 10$ is shown in Fig. 4. It is clear that, PDF gives a faster response with smaller M_p and therefore, for the criterion used it is considered superior to the optimal controller.

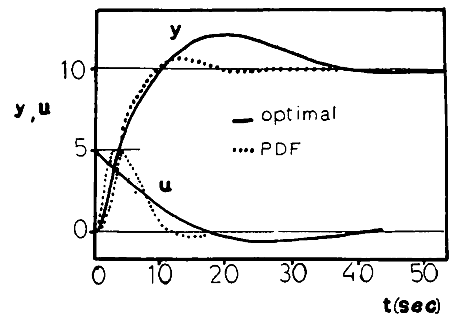


Figure 4. The responses $y(t)$ for the case of equal u_{max}/r and ζ.

Case 2. *Equal* u_{max}/r *and* M_p $(L=0)$

Let $u_{max}/r = 0{\cdot}5$ and $M_p = 4{\cdot}3\%$. Using the results of Fig. 2 and by simulation, one can determine the various parameter values. The results are summarized in the Table. Figure 5 shows that PDF control gives a faster response ($t_{p,\mathrm{PDF}} = 4{\cdot}4$ s and $t_{p,\mathrm{OPT}} = 12{\cdot}3$ s).

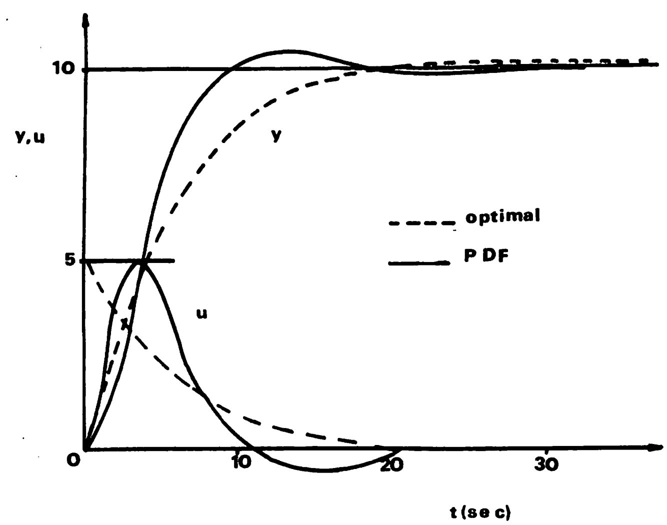


Figure 5. The responses $y(t)$ for the case of equal u_{max}/r and M_p.

Case 3. *Equal* M_p, t_r

Here, t_r is the rise time required for the response to go from $y = 0{\cdot}1r$ to $y = 0{\cdot}9r$. Let $t_r = 1$ s and $M_p = 4{\cdot}3\%$. Results for this case are contained in

Case	Criterion used	Type of control	ζ	ω_n rad/s	K_i/K_2 as applicable	K_{d_1}/K_1 as applicable	ρ	q	Figure	Best control law
1	$u_{max}/r=0{\cdot}5$	PDF	0·707	0·365	0·399	1·548	—	—	4	PDF
	$\zeta=0{\cdot}707$	OPT	0·707	0·118	0·042	0·500	0·0	0·00174		
2	$u_{max}/r=0{\cdot}5$	PDF	0·707	0·365	0·399	1·548	—	—	5	PDF
	$M_p=4{\cdot}3\%$	OPT	2·150	0·039	$4{\cdot}5\times10^{-3}$	0·500	0·22	2×10^{-5}		
3	$t_r=1$ s	PDF	0·707	1·800	0·770	2·160	—	—	6	PDF
	$M_p=4{\cdot}3\%$	OPT	2·150	0·365	4·710	0·400	19·8	0·16		
4	$\zeta=1{\cdot}0$	PDF	1·000	0·453	0·616	2·720	—	—	7	PDF
	$\omega_n=0{\cdot}453$	OPT	1·000	0·453	0·616	2·720	3·70	0·38		

the Table. Figure 6 shows that PDF gives a smaller settling time. Moreover, PDF reaches a maximum control signal of $u_{max}/r = 2{\cdot}5$, whereas the optimal controller reaches $u_{max}/r = 4{\cdot}6$.

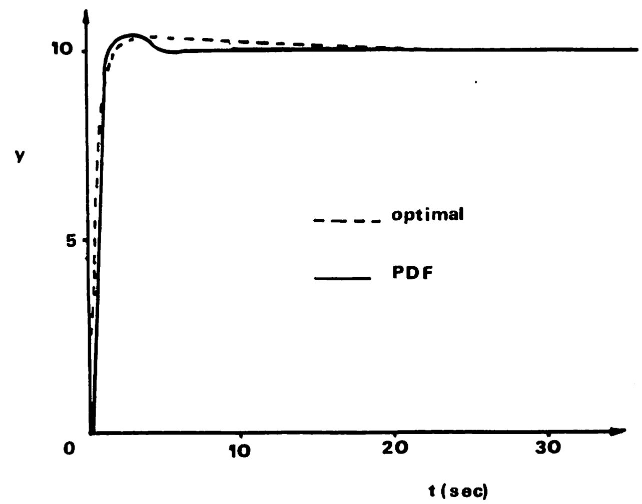


Figure 6. The responses $y(t)$ for the case of equal M_p and t_r.

Case 4

Here the parameters are chosen to have exactly the same response to a step load $L = 10$ while $r = 0$. For example, by choosing $K_1 = K_{d_1} = 2{\cdot}72$ and

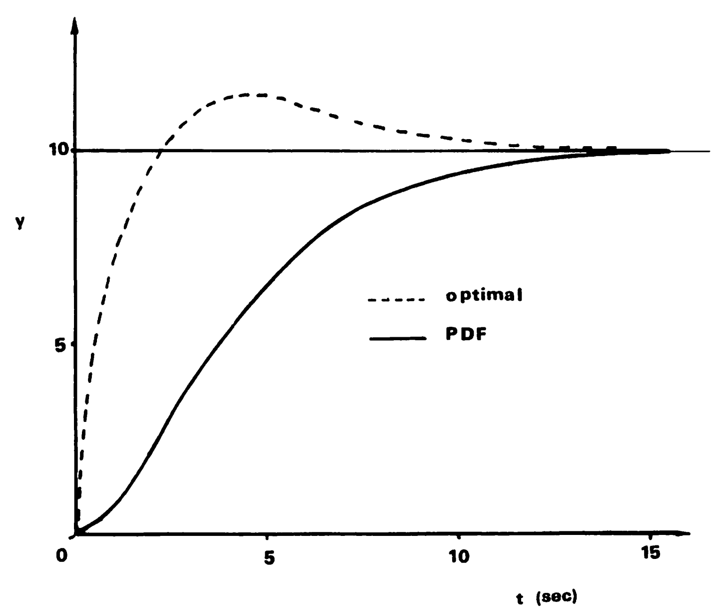


Figure 7. The responses $y(t)$ for the case of equal feedback coefficients.

$K_2 = K_i = 0{\cdot}616$, one can obtain all the system parameters. With the controller gains chosen equal, the system is tested for a step reference input $r = 10$, with $L = 0$. The two responses of Fig. 7 indicate that the optimal controller gives a faster response but undergoes a larger overshoot. In addition, $u_{\max}/r = 2{\cdot}7$ for the optimal and $u_{\max}/r = 0{\cdot}5$ for the PDF.

Cases 1–4 are summarized in the Table. The PDF control is considered to be superior.

From a control input point of view, the two expressions are

$$u_{\mathrm{PDF}} = K_{d_1} y + K_i \int_0^t e\, dt$$

$$u_{\mathrm{OPT}} = K_1 e + K_2 \int_0^t e\, dt$$

Supposing that at $t = 0$, $y = 0$, it is clear that at $t = 0$, $u_{\mathrm{PDF}} = 0$ and $u_{\mathrm{OPT}} = K_1 r$. This means, that the control input u_{PDF} varies between 0 and u_{ss}, whereas u_{OPT} varies between $K_1 r$ and u_{ss}. Thus, PDF uses the control power more efficiently than the optimal controller (Fig. 8).

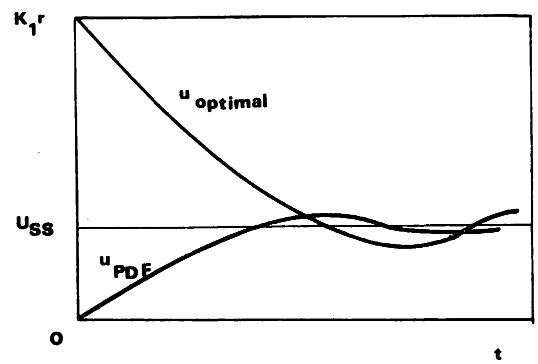


Figure 8. The control inputs $u_{\mathrm{PDF}}(t)$ and $u_{\mathrm{OPT}}(t)$ for comparable characteristics.

6. Development of a new controller

The results of the previous section appear to confirm, that PDF control possesses attractive features, so that one is led to the natural question, as to whether it is possible to apply PDF control to every system.

The implementation of the PDF control scheme has two serious limitations. Firstly, with today's technology, the generation of high order signal derivatives is avoided, because of detrimental noise effects. Secondly, PDF control cannot be used effectively in systems that have more than one open loop zeros. This is due to the fact that, in a system with k zeros, only $(n+1)-k-1$ derivatives, constitute an admissible set of state variables (Brockett 1965). These derivatives, along with the output y and the integral of the error are fed back. So, to be able to place the $(n+1)$ poles of the control system, k has to be no greater than 1.

In spite of the fact that PDF is not widely applicable, it demonstrates very good performance characteristics when utilized with certain low order plants.

In § 5, the optimal control law was seen to introduce a closed-loop zero, that caused a performance degradation. It can be shown that this feature is inherent to this law. What is needed is an optimal control law, which does not add closed-loop zeros. In summary, the control law sought must obey the following rules :

(*a*) it is applicable to every system ;
(*b*) it contains integral action ; and
(*c*) it does not introduce closed loop zeros.

The key to the successful design is the proper selection of the cost functional. The optimal cost must be finite in order to be optimal. Therefore, each penalized term must go to zero when time tends to infinity. The vector $\mathbf{x}-\mathbf{x}_{ss}$ and the variable $u-u_{ss}$ possess this property. However, the magnitude of u_{ss} required to make $e_{ss}=0$ is not known. Suppose now, an integrator is introduced, whose input is the error e. The output of this integrator is bounded (stable system), and as time tends to infinity, the output tends to a constant value. The following problem formulation actually uses this output to produce u_{ss}.

Consider the nth-order system

$$\dot{\mathbf{x}} = A\mathbf{x} + \mathbf{b}u + \mathbf{L} \tag{24 a}$$

$$y = \mathbf{c}^T\mathbf{x} \tag{24 b}$$

and the performance index

$$J = \int_0^\infty [(\mathbf{x}-\mathbf{x}_{ss})^T Q(\mathbf{x}-\mathbf{x}_{ss}) + \rho(u-u_{ss})^2]\, dt \tag{25}$$

Add the equation

$$\dot{x}_{n+1} = \mathbf{c}^T\mathbf{x} - r = e \tag{26}$$

where r signifies the set point. As a first step formulate the augmented system

$$\begin{bmatrix} \dot{\mathbf{x}} \\ \cdots \\ \dot{x}_{n+1} \end{bmatrix} = \begin{bmatrix} A & \vdots & 0 \\ \cdots & & \cdots \\ \mathbf{c}^T & \vdots & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x} \\ \cdots \\ x_{n+1} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ \cdots \\ 0 \end{bmatrix} u + \begin{bmatrix} 0 \\ \cdots \\ -1 \end{bmatrix} r + \mathbf{L} \tag{27}$$

$$y = [\mathbf{c}^T \;\vdots\; 0] \begin{bmatrix} \mathbf{x} \\ \cdots \\ x_{n+1} \end{bmatrix} \tag{28}$$

From the steady state conditions

$$A\mathbf{x}_{ss} + \mathbf{b}u_{ss} + \mathbf{L} = 0 \tag{29 a}$$

$$\mathbf{c}^T\mathbf{x}_{ss} - r = 0 \tag{29 b}$$

one obtains the steady state expressions

$$u_{ss} = -(\mathbf{c}^T A^{-1}\mathbf{b})^{-1}(r + \mathbf{c}^T A^{-1}\mathbf{L}) \tag{30 a}$$

$$\mathbf{x}_{ss} = -A^{-1}(\mathbf{L} + \mathbf{b}u_{ss}) \tag{30 b}$$

However, the relations (30 a, b) do not specify the final value of the variable x_{n+1}. Denoting by h the unknown steady state value $(x_{n+1})_{ss}$, define the following transformation

$$\mathbf{x}=\mathbf{x}'+\mathbf{x}_{ss} \tag{31 a}$$

$$u=u'+u_{ss} \tag{31 b}$$

$$x_{n+1}=x'_{n+1}+h \tag{31 c}$$

Combining eqns. (27), (29) and (31) yields

$$\begin{bmatrix} \dot{\mathbf{x}}' \\ \dot{x}'_{n+1} \end{bmatrix} = \begin{bmatrix} A & 0 \\ \mathbf{c}^T & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}' \\ x'_{n+1} \end{bmatrix} + \begin{bmatrix} \mathbf{b} \\ 0 \end{bmatrix} u' \tag{32}$$

In general, the new component x'_{n+1} is included in the modified cost functional

$$J'=\int_0^\infty ([\mathbf{x}'^T \mid x'_{n+1}]Q'[\mathbf{x}'^T \mid x'_{n+1}]^T+\rho u'^2)\, dt \tag{33}$$

The optimal solution to the above LQ problem yields the optimal feedback control law

$$u'=u-u_{ss}=-\mathbf{K}^T(\mathbf{x}-\mathbf{x}_{ss})-K_{n+1}(x_{n+1}-h) \tag{34}$$

But h is a free constant, and for convenience it can be chosen to be

$$h=-K_{n+1}^{-1}(\mathbf{K}^T\mathbf{x}_{ss}+u_{ss}) \tag{35}$$

Then, eqn. (34) becomes

$$u=-\mathbf{K}^T\mathbf{x}-K_{n+1}x_{n+1} \tag{36}$$

The above control law drives the output of the integrator, whose input is e, to the value of eqn. (35), i.e. as time tends to infinity the steady state error is driven to zero. This control law is shown in Fig. 9.

With the method just described, the integrator produces the required value u_{ss}, and consequently the system is not susceptible to parameter changes. Also, this control scheme has the ability to eliminate the load effects and to adjust its output to obtain $y_{ss}=r$. In addition, it can easily be shown that the law of eqn. (36) does not introduce uncontrollable closed-loop zeros. In summary, the control law developed satisfies all the conditions posed at the beginning of this section and in turn all the specifications stated in § 1.

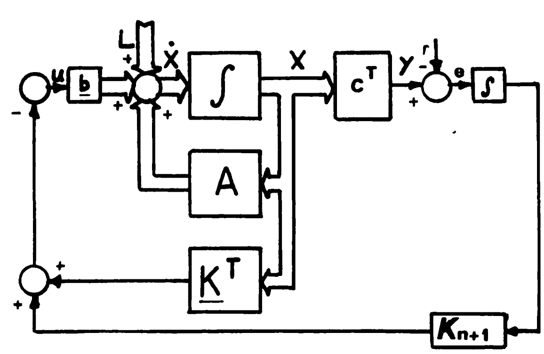


Figure 9. Realization of the control law derived herein.

Furthermore, if it is assumed that $\mathbf{x}(0)=0$ and $x_{n+1}(0)=0$, then $u(0)=0$, i.e. the control law described will change the control input u gradually, avoiding sudden changes.

In the following section this design method is illustrated via a second-order system and it is compared to other control methods.

7. Comparison of PDF, PID and optimal control law

In PDF control, the input u can be expressed as

$$u_{\mathrm{PDF}} = -K_{d_1} y \dots - K_{d_n} y^{(n-1)} - K_i \int_0^t e\,dt \tag{37}$$

The optimal control law of § 6 is given by

$$u = -K_1 x_1 \dots - K_n x_n - K_{n+1} \int_0^t e\,dt \tag{38}$$

When the system is represented by phase variables and $y=x_1$, the two laws have the same structure. The realization of the PDF law requires two sensors and some electronic hardware, whereas the optimal control law requires $n+1$ sensors. Ideally, each law can place the closed-loop poles to arbitrary positions.

However, the PDF controller is not effective for high order systems and also the selection of the feedback coefficients has to be made primarily by trial and error. On the other hand, in the optimal controller design, the feedback gains are determined by a straightforward procedure, and for this, one needs to define the relative importance of the various design goals, e.g. small error and fast time response. Thus, since the optimal control theory gives rise to controllers at least as good as the PDF ones, in the example to follow attention will be focused to the optimal controller developed in § 6.

Suppose the task is to design a controller for a sluggish second-order system. The option of a PID (three term controller) will be compared with the optimal controller. Let the transfer function of the plant be

$$G_p(s) = \frac{1}{s^2 + a_1 s + a_0} \tag{39}$$

with $a_1=5$ and $a_0=6$. By using the well known Ziegler–Nichols (1942) procedures, the resulting control input for a PID controller is

$$u_{\mathrm{PID}}(t) = -81 \cdot 8e - 4 \cdot 0\dot{y} - 409 \cdot 0 \int_0^t e\,dt \tag{40}$$

Using the Cohen–Coon method, from Shih and Chen (1974), the resulting control input for a PID controller is found to be very close to that of eqn. (40).

To derive the optimal controller, rewrite the system equations

$$\dot{x}_1 = x_2 \tag{41 a}$$

$$\dot{x}_2 = -a_0 x_1 - a_1 x_2 + u \tag{41 b}$$

$$\dot{x}_3 = x_1 - r = y - r = e \tag{41 c}$$

and the cost functional

$$J = \int_0^\infty \rho(x_1 - x_{1,ss})^2 + p(x_2 - x_{2,ss})^2 + q(x_3 - x_{3,ss})^2 + (u - u_{ss})^2 \, dt \tag{42}$$

For a given set of weighting coefficients ρ, p, q, the optimal feedback gains can be computed by solving the algebraic Riccati equation. A usual way to define ρ, p, q is as follows

$$\rho = (u_{\max}/e_{\max})^2 \tag{43 a}$$

$$p = (u_{\max}/\dot{y}_{\max})^2 \tag{43 b}$$

$$q = (u_{\max}/\int e \, dt)^2 \tag{43 c}$$

where, $u_{\max}$ corresponds to a maximum desirable input u. Assuming that $r = 10$, it is found from eqn. (30 a) that $u_{ss} = 6r = 60$. Therefore, one may choose $u_{\max} = 100$. Since the minimization of the error e and its integral are of primary interest, one may choose small maximum values, e.g. $e_{\max} = 1$, $\int e \, dt_{\max} = 1$. Since a fast response is desired, the derivative $\dot{y}$ should have a large value, e.g. pick $\dot{y}_{\max} = 10$. These selections yield

$$\rho = 10\,000 \quad p = 100 \quad q = 10\,000 \tag{44}$$

The resulting optimal control law of § 6 is

$$u(\mathbf{x}) = -45{\cdot}6y - 9{\cdot}7\dot{y} - 100 \int_0^t e \, dt \tag{45}$$

It is clear that, the last design procedure not only specifies directly the feedback coefficients, but suggests also the way to improve system performance. For example, to achieve a faster response, $\dot{y}_{\max}$ and $u_{\max}$ should be increased while $e_{\max}$ and $\int e \, dt_{\max}$ should be decreased.

An alternative way to derive the feedback coefficients is to use a pole placement technique, e.g. Karnopp (1976). For the system of eqn. (39) the closed loop characteristic equation is

$$s^3 + (a_1 + K_2)s^2 + (a_0 + K_1)s + K_3 = 0 \tag{46}$$

Suppose the response of this third-order system should resemble that of a second-order system with $\zeta = 0{\cdot}707$ and $t_{\text{peak}} = 0{\cdot}8$ s. This implies $\omega_n = 5$ rad/s, so the pair of the complex conjugate poles is $s_{1,2} = -3{\cdot}5 \pm j3{\cdot}5$. The third pole is placed at -15, to have little impact on the response. Through some algebra, the feedback law, corresponding to these poles is found to be

$$u(\mathbf{x}) = -231{\cdot}1y - 24{\cdot}1\dot{y} - 375 \int_0^t e \, dt \tag{47}$$

This law would have been derived from an optimal control problem having cost functional weighting coefficients

$$\rho = 2{\cdot}11 \times 10^5, \quad p = 362, \quad q = 140\,625 \tag{48}$$

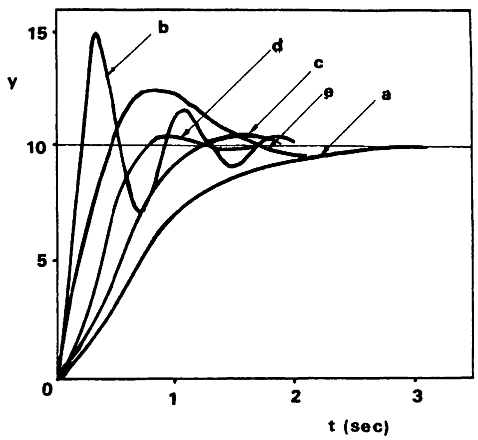


Figure 10. Step responses of plant (39) with control laws: (*a*) open loop control, $u=60$: (*b*) PID control, eqn. (40) : (*c*) optimal control, eqn. (45) ; (*d*) optimal control, eqn. (47) ; and (*e*) optimal control with closed loop zero, $u=-45{\cdot}6e-9{\cdot}7\dot{y}-100\int e\,dt$.

Figure 10 depicts the time responses of the plant given by eqn. (39), with different control laws. It is clear that the optimal control laws (*c*) and (*d*) achieve the best responses. On the other hand, the PID law (40) corresponds to a negative coefficient $\rho=-106$, which means that this law could not have been derived from optimal control theory (see also Shih and Chen 1974).

8. High order systems

As discussed earlier, the PDF scheme cannot be used to control high order systems. To bypass this limitation, Phelan (1977) suggested two alternatives : either ' to redesign the system so that it can more closely be approximated as a second-order system, or split the controlled system into parts, such that each part can be treated as an independent second-order (or lower) system '.

The first approach attempts to solve the problem by ignoring its nature, therefore it will not be considered further. The second approach leads to several subsystems controlled by individual PDF controllers tuned separately. Consider, for example, the fourth-order system of Fig. 11.

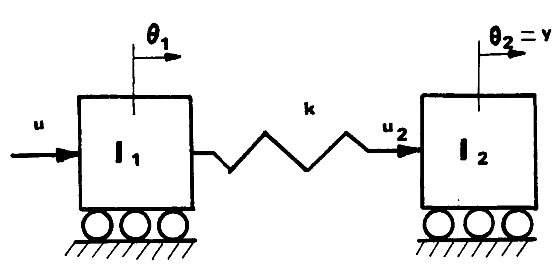


Figure 11. A fourth order system.

Splitting the system into two subsystems and applying Newton's second law, one obtains for the second subsystem

$$\frac{y}{u_2}=\frac{1}{I_2 s^2} \tag{49}$$

The transfer function of the whole system is

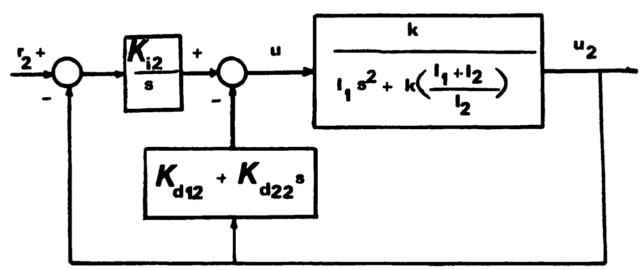
$$G(s)=\frac{y}{u}=\frac{K}{I_1 I_2 s^4+K(I_1+I_2)s^2} \tag{50}$$

Thus, the first subsystem is

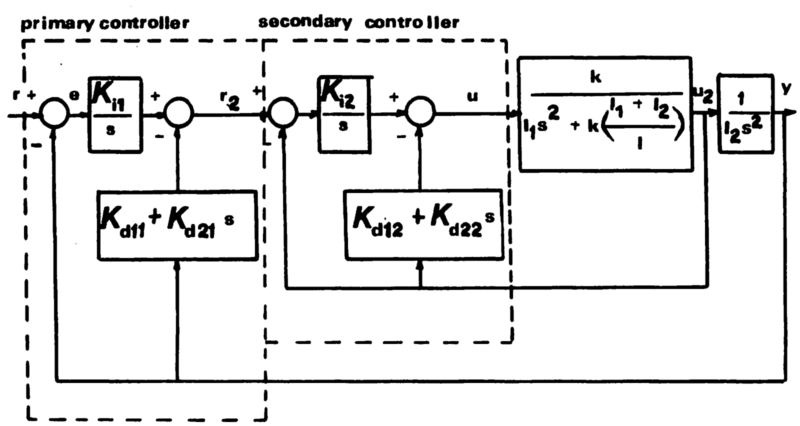
$$\frac{u_2}{u}=\frac{K}{I_1 s^2+K(I_1+I_2)/I_2} \tag{51}$$

The two subsystems (49) and (51) are controlled by separate PDF controllers as shown in Fig. 12.

It is seen from Fig. 12 (*b*) that the final system is of sixth-order. Hence, six controller parameters have to be tuned by trial and error procedures. The overall control scheme is no longer an output controller, but resembles a modern state feedback controller.



(*a*)



(*b*)

Figure 12. (*a*) PDF control of subsystem (51); (*b*) two PDF controllers ' in series ' are used to control complete system (50).

An alternative way to control the system (50) is by employing the control law of § 6. Using as state variables the two velocities and the two positions, the optimal control law, determined in a straightforward manner, leads to a fifth-order system. This scheme requires an integrator less and two more sensors. Clearly, this controller is much more flexible, directly tunable and therefore preferable to use.

9. Conclusions

The object of this paper has been to consider the control problem of single-input–single-output systems whose performance requires: (*a*) zero steady state error; (*b*) elimination of load effects from the output; and (*c*) immunity to parameter variables while conditions (*a*) and (*b*) are valid.

The achieve these goals, an integration of the error signal was introduced and the optimal control theory was applied using a reformulated cost functional.

Previous attempts to face this problem, viz. PDF control, are deemed unsatisfactory. The optimal controller proposed herein overcomes the drawbacks of previous designs and is shown to be better than PDF and PID controllers, because: (*a*) it can be used to control any system; (*b*) it assures stability; and, (*c*) the feedback gains are amenable to optimization and can be determined by straightforward techniques.

The design procedures, based on the optimal control theory, can generate controllers superior than the classical ones, provided the respective problem is well formulated. This statement, of course, does not imply that optimal controllers should be implemented on all occasions, but rather they should be included in the candidate list of engineering-economic trade-off feasibility studies.

References

Athans, M., 1971, *Automatica*, **7**, 643.

Brockett, R. W., 1965, *I.E.E.E. Trans. autom. Control*, **10**, 129.

Hedrick, J. K., 1978, *A Note on Using Error Coordinates as State Variables*, Lecture Notes, M.I.T.

Johnson, C. D., 1968, *I.E.E.E. Trans. autom. Control*, **13**, 416; 1970, *Ibid.*, **15**, 222.

Karnopp, D., 1976, *Designing Low-Order Near Optimum Controllers with just a Little Algebra*, A.S.M.E. Winter Annual Meeting (New York).

Kwakernaak, H., and Sivan, R., 1972, *Linear Optimal Control Systems* (Wiley–Interscience).

Phelan, R. M., 1977, *Automatic Control Systems* (Ithaca, London: Cornell University Press).

Shih, Y. P., and Chen, C. J., 1974, *Int. J. Control*, **19**, 947.

Ziegler, J. G., and Nichols, N. B., 1942, *Trans. Am. Soc. mech. Engrs*, **64**, 759.